The rank of a matrix.

We present the rank of a matrix in a way different from the textbook. We can give a more precise statement of the rank theorem using RREF.

**Def.** \( A \in M_{m \times n}(F) \).

Recall \( L_A : F^n \rightarrow F^m, L_A(x) = Ax \).

**Def.** \( \text{rank}(A) = \text{rank}(L_A) \).

**Def.** \( A \in M_{m \times n}(F) \).

\( \text{Col}(A) = \text{span}\{\text{columns of } A\} \subseteq F^m \)

\( \text{Row}(A) = \text{span}\{\text{rows of } A\} \subseteq F^n \)

\( \text{Col}(A) \) and \( \text{Row}(A) \) are subspaces of \( F^m \) and \( F^n \) resp.

**Thm 3.5** \( \text{rank}(A) = \dim \text{Col}(A) \).

**Proof.** Check the definition. \( \square \)
Thm (Rank Theorem).

Let $A \in M_{m \times n}(F)$ have RREF $B$.

(a) Suppose $B$ has $r$ nonzero rows $R_1, \ldots, R_r$. Then $\dim(\text{Row}(A)) = r$, and $R_1, \ldots, R_r$ form a basis of $\text{Row}(A)$.

(b) Let $i_1, \ldots, i_r$ be the numbers of the pivot columns of $B$, hence $A$. Then $\dim(\text{Col}(A)) = r$, and the pivot columns of $A = (a_1, \ldots, a_n)$, $a_{i_1}, \ldots, a_{i_r}$, form a basis of $\text{Col}(A)$.

(c) $\text{rank}(A) = r$.

Cor. $\dim(\text{Row}(A)) = \dim(\text{Col}(A)) = \text{rank}(A)$
Proof.

(a) Since \( A \) and \( B \) are row equivalent (that is, they are obtained from one another by elementary row operations), \( \text{Row} \ (A) = \text{Row} \ (B) \).

Obviously, \( \text{Row} \ (B) = \text{span} \{ R_1, \ldots, R_r \} \) because the other rows are zeros.

The rows \( R_1, \ldots, R_r \) are linearly independent because the leading units are the only nonzero entries in their columns.

Indeed, if \( x_1 R_1 + \ldots + x_r R_r = 0 \), then the slots \( i_1, \ldots, i_r \) of the row \( x_1 R_1 + \ldots + x_r R_r \) will contain exactly \( x_1, \ldots, x_r \). Hence \( x_1 = \ldots = x_r = 0 \).

Now \( R_1, \ldots, R_r \) form a basis of \( \text{Row} \ (A) = \text{Row} \ (B) \), and \( \dim \text{Row} \ (A) = r \).
(b) Since $A$ and $B$ are row equivalent, the equations $Ax = 0$ and $Bx = 0$ are equivalent. In other words, using the notation $A = (a_1, \ldots, a_n)$, $B = (b_1, \ldots, b_n)$, $x = \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right)$,

$x_1 a_1 + \ldots + x_n a_n = 0$ if and only if $x_1 b_1 + \ldots + x_n b_n = 0$.

This means that if a certain subset of $\{b_1, \ldots, b_n\}$ is linear independent, then the corresponding subset of $\{a_1, \ldots, a_n\}$ is linearly independent. If a certain subset of $\{b_1, \ldots, b_n\}$ generates $\text{Col}(B)$, then the corresponding subset of $\{a_1, \ldots, a_n\}$ generates $\text{Col}(A)$.

By the structure of $RREF$, the pivot columns $b_{i_1}, \ldots, b_{i_r}$ are in fact $e_1, \ldots, e_r$, the vectors
from the standard basis \( \{e_1, \ldots, e_m\} \) of \( F^m \). Clearly \( e_1, \ldots, e_r \) are linearly independent and form a basis of \( \text{Col}(B) = \{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ x_{r+1} = \ldots = x_m = 0 \end{pmatrix} \} \).

Hence, the corresponding pivot columns \( a_{i_1}, \ldots, a_{i_r} \) form a basis of \( \text{Col}(A) \), and \( \dim \text{Col}(A) = r \).

(c) \( \text{rank}(A) = \dim \text{Col}(A) = r \).

Remark. In the above proof, \( \dim \text{Col}(A) = \dim \text{Col}(B) = r \). However, \( \text{Col}(A) \neq \text{Col}(B) \) in general. We will have \( \text{Col}(A) = \text{Col}(B) \) only if all rows of \( A \) with numbers above \( r \) are zeros. (Prove!)

In particular, if \( r = m \), then \( \text{Col}(A) = \text{Col}(B) = F^m \).