2.3. Composition and Matrix Multiplication

We present this material in a different order than in the textbook.

Understanding the matrix of a linear transformation.

Thm (2.14) Let $T: V \rightarrow W$ be a linear transformation. Let $\beta$ and $\gamma$ be finite bases of $V$ and $W$ respectively. Then for all $v \in V$ we have $[T(v)]_\gamma = [T]_\beta^{\gamma} [v]_\beta$.

Proof. We give a direct proof.

(The elegant proof in the textbook uses Thm 2.11. I think that proof is hard to understand.)

Let $\beta = \{v_1, \ldots, v_n\}$.

Let $v \in V$, let $x = [v]_\beta = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$.

Then $v = x_1 v_1 + \ldots + x_n v_n$. 


Since $T$ is linear,

$$T(v) = x_1 T(v_1) + \ldots + x_n T(v_n).$$

Recall $\Psi_y : W \rightarrow \mathbb{R}^m$, $\Psi_y(v) = [v]_y$.

Since $\Psi_y$ is linear,

$$[T(v)]_y = x_1 [T(v_1)]_y + \ldots + x_n [T(v_n)]_y$$

$$= ([T(v_1)]_y \ldots [T(v_n)]_y) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= [T]_y^T [v]_\beta.$$  \[ \square \]

By Thm 2.14, the following diagram is commutative (that is, vertical arrows commute with horizontal arrows)

$$
\begin{array}{ccc}
V & \xrightarrow{T} & W \\
\downarrow \Psi_\beta & & \downarrow \Psi_y \\
\mathbb{R}^n & \xrightarrow{L_A} & \mathbb{R}^m \\
& & L_A(x) = Ax.
\end{array}
$$

See Example 3 in the textbook.
Cor. (Thm 2.15d) Let \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation. Then there is a matrix \( A \in M_{m \times n}(\mathbb{R}) \) such that \( T(v) = Av \).

**Proof.** Let \( \beta \) and \( \gamma \) be the standard bases of \( \mathbb{R}^n \) and \( \mathbb{R}^m \) resp. Since the bases are standard, \( \forall v \in V, \ [v]_\beta = v, \ [T(v)]_\gamma = T(v) \). Put \( A = [T]_\beta^\gamma \).

By Thm 2.14, \( \forall v \in V, \ T(v) = [T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta = Av. \)
Composition of linear transformations

Def. Let $T : V \rightarrow W$ and $U : W \rightarrow Z$ be functions. Then $UT : V \rightarrow Z$ is a function defined by $(UT)(v) = U(T(v))$. $UT$ is the composition of $U$ and $T$.

Thm. (2.9) Let $T : V \rightarrow W$ and $U : W \rightarrow Z$ be linear transformations. Then $UT : V \rightarrow Z$ is linear.

Proof. Obvious. □

Define $I_V : V \rightarrow V$, the identity transformation of $V$, by $I_V(x) = x$, $x \in V$.

Sometimes we just write $I$ omitting $V$. 
Properties of the composition

Thm (2.10).

Omitting the vector spaces:

\[ T(U_1 + U_2) = TU_1 + TU_2 \]
\[ (T_1 + T_2) U = T_1 U + T_2 U \]
\[ (T_1 T_2) T_3 = T_1 (T_2 T_3) \]
\[ T I = I, \quad I T = I \quad (I \text{ is the identity}) \]

Proof. Easy
Matrix multiplication.

\[ A = (a_{ij}) \in M_{mxn} \]
\[ B = (b_{ij}) \in M_{nxp} \]

Define \( C = AB = (c_{ij}) \in M_{mn} : \)

\[ c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \]

\[ AB = \begin{pmatrix} a_{i1} & \cdots & a_{in} \\ \vdots & \ddots & \vdots \\ a_{im} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix} = \begin{pmatrix} \vdots \\ c_{ij} \end{pmatrix} = C \]

Row-column rule.

Special case: \( p = 1 \) \( M_{nx1} = \mathbb{R}^n \)

\[ A = (a_1 \cdots a_n) \in M_{nxn}, \quad a_i \in \mathbb{R}^m \]

We have already defined for \( x \in \mathbb{R}^n \)

\[ Ax = x_1 a_1 + \cdots + x_n a_n \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \]

This definition agrees with the above.
General case (Thm 2.13)

Let \( A \in M_{m \times n} \)

Let \( B = (b_1, \ldots, b_p) \in M_{n \times p}, \) \( b_i \in \mathbb{R}^n. \)

Then \( AB = (Ab_1, \ldots, Ab_p). \)

Proof. Follows by the above definition.

Properties (Thm 2.12)
(assuming that all products make sense.)

\[ A(B + C) = AB + AC \]

\[ (A + B)C = AC + BC \]

\[ \alpha \in \mathbb{R}, \quad \alpha(AB) = (\alpha A)B = A(\alpha B) \]

\[ IA = A \quad AI = A \]

Here \( I = I_n = \begin{pmatrix} 1 & 0 \\ 0 & I_n \end{pmatrix} = (e_1, \ldots, e_n) \)

is the identity matrix.

Proof. Easy exercise.

(See examples in the textbook.)
**Warning:** $AB \neq BA$ in general.

**Ex.**
\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}
\]
\[
AB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}
\]
\[
BA = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}
\]

**Thm (2.16).** $(AB)C = A(BC)$.

**Proof.** We give a direct proof.

Let $A = (a_{ij}) \in M_{m \times n}$, $B = (b_{ij}) \in M_{n \times p}$, $C = (c_{ij}) \in M_{p \times q}$.

Then $AB \in M_{m \times p}$, $BC \in M_{n \times q}$.

\[
((AB)C)_{ij} = \sum_{l=1}^{p} (AB)_{il} c_{lj} = \sum_{l=1}^{p} (\sum_{k=1}^{n} a_{ik} b_{kl}) c_{lj} = \sum_{k=1}^{n} \sum_{l=1}^{p} a_{ik} b_{kl} c_{lj}
\]

\[
(A(BC))_{ij} = \sum_{k=1}^{n} a_{ik} (BC)_{kj} = \sum_{k=1}^{n} a_{ik} (\sum_{l=1}^{p} b_{kl} c_{lj}) = \sum_{k=1}^{n} \sum_{l=1}^{p} a_{ik} b_{kl} c_{lj}
\]

The results differ only in the order of summation. Hence, they are the same.
Cor. (Thm 2.15e)
\[ L_{AB} = L_A L_B \] (assuming \( AB \) makes sense.)

Proof
\[ L_{AB}(x) = (AB)x \xrightarrow{2.16} A(Bx) = L_A(Bx) \]
\[ = L_A(L_B(x)) = (L_A L_B)(x) \]

Lemma
Let \( A, B \in M_{m \times n} \)
Suppose \( \forall x \in \mathbb{R}^n \quad Ax = Bx \)
Then \( A = B \)
(In other words, if \( L_A = L_B \), then \( A = B \))

Proof
Let \( A = (a_1, \ldots, a_n) \)
\( B = (b_1, \ldots, b_n) \)
Let \( x = e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \) - i
\( \Delta_0 \) gives \( Ae_i = a_i, \quad Be_i = b_i \)
\( Ae_i = Be_i \quad \Rightarrow \quad a_i = b_i \quad \Rightarrow \quad A = B \)
Matrix of a composition.

Thm. (2.11)
Let $T : V \to W$, $U : W \to Z$ be linear transformations. Let $\alpha, \beta, \gamma$ be finite bases of $V, W, Z$ resp. Then

$$[UT]_\gamma^\beta = [U]_\beta^\gamma [T]_\alpha^\beta$$

(Strictly speaking, this theorem is not proven in the textbook. There is only a discussion on p. 87 before a definition of the matrix product.)

Proof. Let $\alpha = \{v_1, \ldots, v_n\}$.

Let $x \in IR^n$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Let $v = \sum_{i=1}^{n} x_i v_i$, $[v]_\alpha = x$.

We have:

$$[(UT)(v)]_\gamma^\beta = [UT]_\alpha^\beta [v]_\alpha = [UT]_\alpha^\beta x$$

Hence the conclusion. See Example 2 in the textbook.