Lecture 1

**Task:** write algorithm to find all solutions of a system of linear equations.

\[ a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \quad E_1 \]
\[ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \quad E_2 \]
\[ \vdots \]
\[ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m \quad E_m \]

**Ex**

\[ x_1 + x_2 = 1 \quad E_1 \]
\[ x_1 + 3x_2 = 2 \quad E_2 \]

\[ E_2 - E_1 \Rightarrow 2x_2 = 1 \quad \text{call this new } E_2 \]

\[ x_1 + x_2 = 1 \quad E_1 \]
\[ 2x_2 = 1 \quad E_2 \]

\[ \frac{1}{2} E_2 \Rightarrow x_2 = \frac{1}{2} \]

Subbing this back in to \( E_1 \), we get \( x_1 = \frac{1}{2} \).

So there is a unique solution, \((x_1, x_2) = (\frac{1}{2}, \frac{1}{2})\).
General Strategy.

i) Manipulate the equations to get to a simpler equivalent system.

ii) Get to a simplest system from which we can derive the solution set.

Three basic equation operations:

1. Swap two equations. $E_i \leftrightarrow E_j$

2. Multiply an equation by a nonzero constant.
   $E_i \rightarrow cE_i \quad (c \neq 0)$

3. $E_i \rightarrow E_i + cE_j \quad (\text{real workhorse})$

Then! None of these operations change the solution set.

PF 1 and 2 obvious.

For 3 note that

\[
\begin{align*}
E_1 + cE_2 & \downarrow \\
\begin{cases}
a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n = b_2
\end{cases} & \Uparrow \\
\begin{cases}
a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\
(a_{21}+ca_{11})x_1 + \cdots + (a_{2n}+ca_{1n})x_n = b_2 + cb_1
\end{cases} & \uparrow
\end{align*}
\]
Observation: in performing 1, 2 and 3 the variables are just placeholders. We can just keep their places.

\[
\begin{align*}
  a_1 x_1 + \ldots + a_n x_n &= b_1 \quad E_1 \\
  &\vdots \quad \vdots \\
  a_m x_1 + \ldots + a_n x_n &= b_m \quad E_m
\end{align*}
\]

\[\rightarrow \begin{pmatrix} a_1 & \cdots & a_n & b_1 \\ \vdots & & \vdots & \vdots \\ a_m & \cdots & a_n & b_m \end{pmatrix} \quad R_n \]

This \( m \times (n+1) \) matrix is the augmented matrix of the system.

The equation operations can be replaced by row operations

1. \( R_i \leftrightarrow R_j \)
2. \( R_i \rightarrow cR_j \)
3. \( R_i \rightarrow R_i + cR_j \)

\[
\begin{align*}
  x_1 + x_2 &= 1 \\
  x_1 + 3x_2 &= 2
\end{align*}
\]

\[
\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}
\]
\[ R_2 \rightarrow \frac{1}{2} R_2 \]
\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & \frac{1}{2}
\end{pmatrix}
\]

\[ \text{LS} \]
\[
x_1 + x_2 = 1
\]
\[
x_2 = \frac{1}{2}
\]

Q. To what form should we try to simplify our linear system (augmented matrix)?

Def. The leading entry of a row is its leftmost nonzero entry.

ex
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 2 & 3 & 4
\end{pmatrix}
\]

Def. A matrix is in Row Echelon Form (REF) if

a) all zero rows are below all nonzero rows.

b) the leading entry of every nonzero row is to the right of the leading entry of the row above.
A matrix is in **reduced row echelon form (RREF)** if it is in RREF and

- Every leading entry is 1.
- Every leading entry is the only nonzero entry in its column.

**Ex.**

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 2 & 1
\end{pmatrix}
\]

not RREF

\[
R_2 \rightarrow \frac{1}{2} R_2
\]

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & \frac{1}{2}
\end{pmatrix}
\]

not RREF

\[
R_1 \rightarrow R_1 - R_2
\]

\[
\begin{pmatrix}
1 & 0 & \frac{1}{2} \\
0 & 1 & \frac{1}{2}
\end{pmatrix}
\]

RREF
Thm 2. Every matrix can be put into RREF by a finite sequence of elementary row operations.

(Gaussian Elimination).

Thm 3. The RREF of a matrix is unique.

(Both our answers should all agree.)

"Thm 4" (Not rigorous) The solution set of a system whose augmented matrix is in RREF is easily described in a standard way.

Let's see that Thm 4 holds in examples.

a) Consider the system by augmented matrix

\[
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
X_1 - X_3 = 0 \\
X_2 = 0 \\
0 = 1
\end{pmatrix}
\]

The last equation is never satisfied so no system has no solutions (we say it is inconsistent).

Moral: leading entry in last column of RREF \iff inconsistent.

b) \[
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
X_1 - X_3 = 0 \\
-X_2 = 0 \\
0 = 0
\end{pmatrix}
\]
This system is consistent and has infinitely many solutions.

Our convention is to parametrize (label) the solutions using the variables corresponding to columns with no leading entries. They are called free variables.

\[
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

corresponds to \( x_3 \)

\( x_3 \) is a free variable

Set \( x_3 = t \) \( E_1 \Rightarrow x_1 = t \) \( E_2 \Rightarrow x_2 = 0 \)

So solutions are \( \{(x_1, x_2, x_3) \mid x_1 = t, x_2 = 0, x_3 = t \} \) \( t \in \mathbb{R} \)

\[
\begin{bmatrix}
1 & 2 & 0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3 & 0 & 3 \\
0 & 0 & 0 & 0 & 1 & 4 \\
\end{bmatrix}
\]

\( x_2 \) and \( x_4 \) are free variables

Set \( x_2 = t_1 \) and \( x_4 = t_2 \)

\( E_1 \Rightarrow x_1 = 2 - 2t_1 - t_2 \)
\( E_2 \Rightarrow x_2 = 3 - 3t_2 \)
\( E_5 \Rightarrow x_3 = 4 \)

Solutions are \( \{(2 - 2t_1 - t_2, t_1, 3 - 3t_2, t_2, 4) \mid t_1, t_2 \in \mathbb{R} \} \)
Slots with leading entries are called **pivot positions** (circled in red).

Columns with leading entries are called **pivot columns**.

Other columns are **non-pivot columns** (circled in green).

Cor. The set of solutions of a consistent linear system depends on $k$ parameters. Here $k$ is the number of **non-pivot columns** (free variables).