1. (5 pts) True or false? (You don’t have to give reasons for your answers.)
   (a) If \( A \in M_{m \times n}(\mathbb{R}) \) and \( \text{rank}(A) = 0 \), then \( A = 0 \).
   
   (b) For two matrices \( A \) and \( B \), \( \text{rank}(AB) \) is the lesser of \( \text{rank}(A) \) and \( \text{rank}(B) \).

   (c) If \( T : V \to W \) is a linear transformation, then \( T \) carries linear independent subsets of \( V \) onto linear independent subsets of \( W \).

   (d) Every change of coordinate matrix is invertible.

   (e) The function \( \det : M_{n \times n}(\mathbb{R}) \to \mathbb{R} \) is a linear transformation.
2. (5 pts) Let $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f) = f'$. Let $U : P_2(\mathbb{R}) \to P_3(\mathbb{R})$, $U(f(t)) = \int^t_0 f(s) \, ds$. Let $\alpha = \{1, t, t^2\}$ and $\beta = \{1, t, t^2, t^3\}$ be the standard bases for $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$ resp. Find $[T]_\beta^\alpha$, $[U]_\alpha^\beta$, and $[UT]_\beta$ directly and verify that $[UT]_\beta = [U]_\alpha^\beta [T]_\beta^\alpha$. 
3. (5 pts) Let $A \in M_{n \times n}(\mathbb{R})$. Recall $L_A : \mathbb{R}^n \to \mathbb{R}^n$, $L_A(x) = Ax$. Prove that $A$ is invertible if and only if $L_A$ is invertible. (You can use that the matrix product is associative.)
4. (5 pts) Let

\[ A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}, \quad \beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}. \]

Find \([L_A]_\beta\) directly and indicate a matrix \(Q\) such that \([L_A]_\beta = Q^{-1}AQ\). (You don’t have to calculate \(Q^{-1}\).)
5. (5 pts) Represent the matrix

\[ A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \]

as a product of elementary matrices.
6. (5 pts)

(a) Let $T$ and $U$ be linear transformations such that the composition $UT$ makes sense. Prove that $\text{rank}(UT) \leq \text{rank}(T)$.

(b) Let $A$ and $B$ be matrices such that the product $AB$ makes sense. Prove that $\text{rank}(AB) \leq \text{rank}(B)$.
7. (5 pts)

(a) Let \( A \in M_{5 \times 5}(\mathbb{R}) \). Then for every \( c \in \mathbb{R} \) we have \( \det(cA) = b \det(A) \) for some \( b \in \mathbb{R} \). Find \( b \) in terms of \( c \).

(b) Let \( A, B \in M_{5 \times 5}(\mathbb{R}) \) and let \( AB = -BA \). Prove that either \( A \) or \( B \) is not invertible. (Hint. Use that \( \det(AB) = \det(A) \det(B) \).)