Distal and Non-Distal Pairs

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Throughout all theories will be taken to be complete.

Given a theory $T$ in language $L$, $\mathcal{M}$ will always refer to a monster model for $T$. That is a model that is sufficiently saturated and sufficiently homogeneous.

For a tuple $a \in \mathcal{M}^p$ and $A \subseteq \mathcal{M}$, by $tp(a|A)$ we mean the collection of all $L_A$ formulas $\varphi(x_1, \ldots, x_p)$ such that $\mathcal{M} \models \varphi(a)$.

For $A \subseteq \mathcal{M}$, by $dcl_{\mathcal{L}}(A)$ we mean the $\mathcal{L}$-definable closure of the set $A$ in $\mathcal{M}$. If $\mathcal{M}$ is o-minimal, this is a pregeometry.
The notion of NIP is one notion of tameness. We give a definition in terms of indiscernible sequences.

**Definition (NIP)**

We call an \( \mathcal{L} \)-formula \( \varphi(x, y) \) dependent (in \( T \)) if for every indiscernible sequence \( (a_i)_{i \in \omega} \) from \( M^p \) and every \( b \in M^q \), there is \( i_0 \in \omega \) such that either \( M \models \varphi(a_i, b) \) for every \( i > i_0 \) or \( M \models \neg \varphi(a_i, b) \) for every \( i > i_0 \). The theory \( T \) is NIP (or is dependent) if every \( \mathcal{L} \)-formula is dependent in \( T \).
The following is one definition of distality for a theory $T$. There are also definitions of distality of a type and distality of an indiscernible sequence.

**Definition (Distality of $T$)**

We say $T$ is *distal* if whenever $A \subseteq \mathbb{M}$, and $(a_i)_{i \in I}$ an indiscernible sequence from $\mathbb{M}^p$ such that

a. $I = I_1 + (c) + I_2$, and both $I_1$ and $I_2$ are infinite without endpoints,

b. $(a_i)_{i \in I_1 + I_2}$ is $A$-indiscernible,

then $(a_i)_{i \in I}$ is $A$-indiscernible.
Famous (Non)Examples

- Algebraically closed fields (of any characteristic) are non-distal.
- In fact, any stable theory is non-distal.
- o-minimal structures are distal.
- Algebraically closed valued fields (ACVF) are non-distal.
Theorem (Hieronymi, N., 2016)

Let $A = (A, +, \ldots)$ be an o-minimal expansion of an ordered group and $B \subseteq A$. Then consider the following pairs $(A, B)$:

1. $(\mathbb{R}, +, \times, 0, 1, <, 2\mathbb{Z})$ (discrete)
2. $A$ expanding a real closed field, $B$ a proper elementary substructure of $A$ such that there is a unique standard part map from $A$ into $B$. (tame pairs)
3. $B$ a proper, dense elementary substructure. (dense pairs)
4. $(\mathbb{R}, +, \times, 0, 1, <, 2\mathbb{Q})$ (dense multiplicative subgroup)
5. $B$ is a dense, definably independent set.

Then (1) and (2) are distal, while (3), (4), and (5) are not.
All of these examples are NIP.

Note that for all of our distal examples are expansions by discrete predicates, while all of the non-distal expansions are dense.

The group structure is essential as \((\mathbb{R}, <, \mathbb{Q})\) is distal.
We show distality by a technical lemma for expansions of o-minimal structures by a single function symbol.

Notice that \((\mathbb{R}, +, \times, 0, 1, <, 2^\mathbb{Z})\) defines the same sets as \((\mathbb{R}, +, \times, 0, 1, <, \lambda)\) where \(\lambda(x) = \min([0, x] \cap 2^\mathbb{Z})\) for \(x > 0\) and 0 otherwise.

The lemma is proven by an induction on the number of occurrences of the function symbol and a localized notion of distality for a single formula.

For the case of tame pairs, they are weakly o-minimal, and thus dp-minimal. Thus by work of Simon are distal.
Let $T$ be a complete $\mathcal{L}$-theory expanding that of ordered groups and $U$ a new unary predicate for a subset $U(\mathbb{M})$ of a monster model for $T$.

In an ordered structure $\mathbb{M}$, $S \subset \mathbb{M}$ is called *small* if there is no $n \in \mathbb{N}$ and a definable function $f : \mathbb{M}^n \to \mathbb{M}$ such that $f(S^n)$ has interior (in the order topology).
The Non-Distal Pairs

Theorem (Hieronymi, N., 2016)

Suppose the following conditions hold:

1. $U(M)$ is small and dense in $M$.
2. For $n \in \mathbb{N}$, $C \subseteq M$, and $a, b \in M^n$ both $dcl_L$-independent over $C \cup U(M)$,

   $$tp_L(a|C) = tp_L(b|C) \Rightarrow tp_L(U)(a|C) = tp_L(U)(b|C).$$

Then $T_U$ is not distal.
Proof.

First take $b$ to be $\text{dcl}_L$-independent over $U(M)$. Then realize the (partial) type in variables $(x_i : i \in I_1 + (c) + I_2)$ expressing the following properties:

(i) $\{x_i : i \in I_1 + I_2\}$ is $\text{dcl}_L$-independent over $U(M)b$,

(ii) $f(x_{i_1}, \ldots, x_{i_n}, b) < x_{i_{n+1}}$, for each $i_1 < \cdots < i_{n+1} \in I_1 + (c) + I_2$ and $L$-$\emptyset$-definable function $f$,

(iii) there is $u \in U(M)$ such that $x_c = u + b$.

A realization $(a_i : i \in I_1 + (c) + I_2)$ will be indiscernible over the empty set. $(a_i : i \in I_1 + I_2)$ will be $b$-indiscernible, but the full sequence is not $b$-indiscernible.
Often the “stable behavior” in a non-distal structure $\mathcal{M}$ occurs in $\mathcal{M}^{eq}$.

In the case of ACVF, there are “generically stable” types that in a certain sense isolate the stable behavior.

This manifests as follows. Given $(a_i)_{i \in I}$ an indiscernible sequence in the valued field sort, $(a_i)_{i \in I}$ is not distal if and only if $v(a_i - a_j) = v(a_{i'} - a_{j'})$ for $i \neq j$ and $i' \neq j'$.
(ℝ, +, 0, <, ℚ)

- By results of van den Dries, this satisfies all of the assumptions of the previous theorem.
- If we consider the quotient \( \mathbb{R}/\mathbb{Q} \), this has induced structure of equality with one constant, which is stable.
- I hope to show that this structure has no good decomposition as in ACVF by showing that this is the “only” imaginary sort.
- Generically stable types in the sort \( \mathbb{R}/\mathbb{Q} \) are not enough to isolate the non-distality.
Here we take $H$ to be a $\text{dcl}_\mathcal{L}$ basis for $\mathbb{R}$. By results of Dolich, Miller, and Steinhorn this has the assumptions of our non-distality theorem.

However, this structure eliminates imaginaries, so there are no imaginary sorts demonstrating the non-distality.

We also fail to get a description as in ACVF, but for a different reason. There simply are no generically stable types, even after passing to $\mathbb{M}^{eq}$. 
Distality, while not preserved under reducts, has some strong consequences that are preserved under reducts.

Chernikov and Starchenko have shown that definable bi-partate graphs in distal theories have the “Strong Erdős-Hajnal” property, a form of combinatorial regularity.

Thus if one can demonstrate that a non-distal structure admits a distal expansion, then this combinatorial regularity is displayed by the original structure.

