Sum-paintability of graphs

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Joint work with
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Def. (Isaak [2002]) The sum-choosability $sc(G)$ is the least $\sum_{v \in V(G)} f(v)$ among f such that G is f-choosable.

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Worst-case analysis is modeled by the following game:

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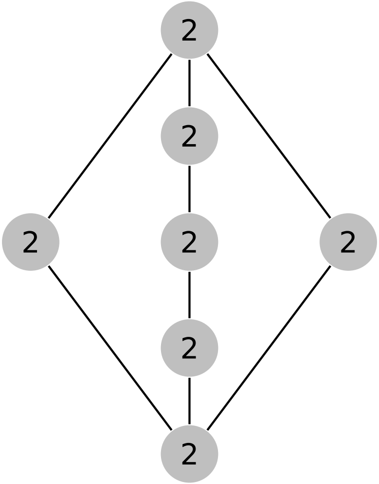
Goal: Lister wins by presenting a vertex with no tokens. Painter wins by coloring all vertices in the graph.

Example Game

Let’s play the Lister/Painter game on $\Theta_{2,2,4}$.



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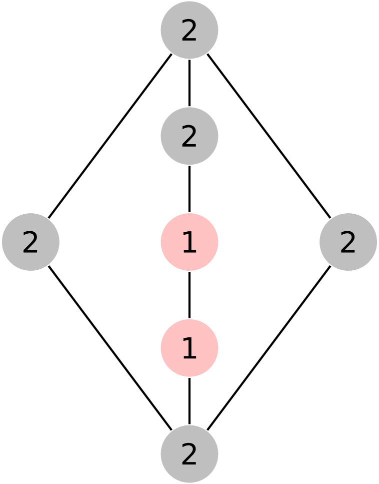

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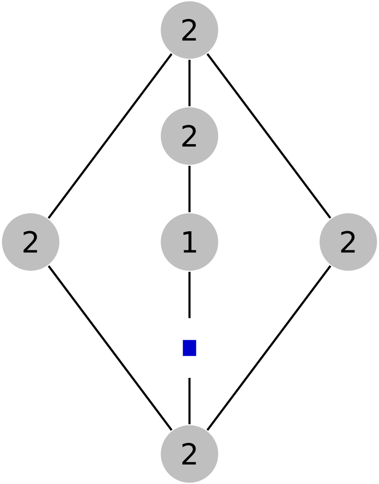

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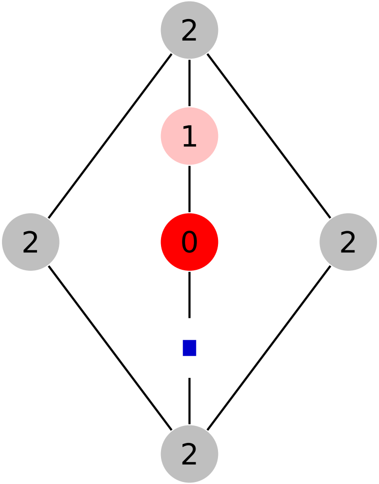

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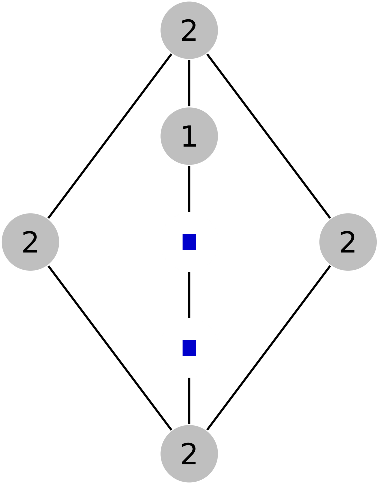

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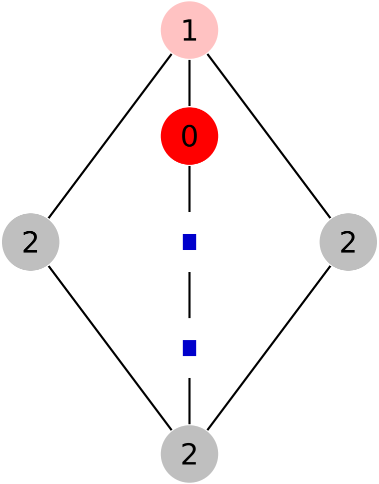

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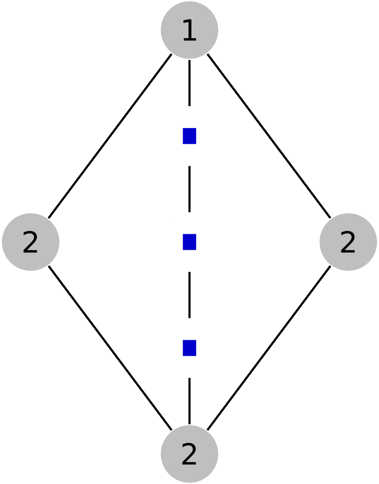

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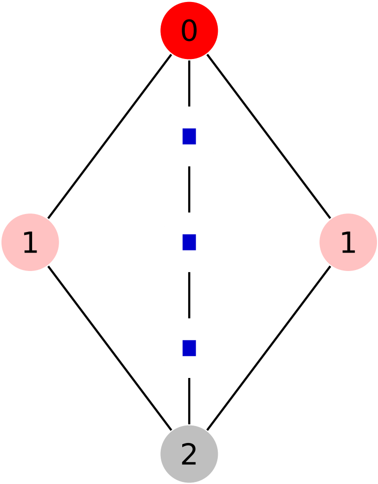

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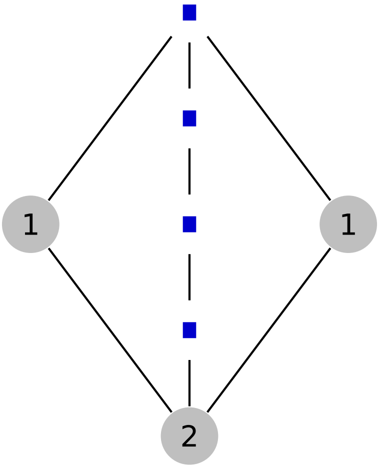

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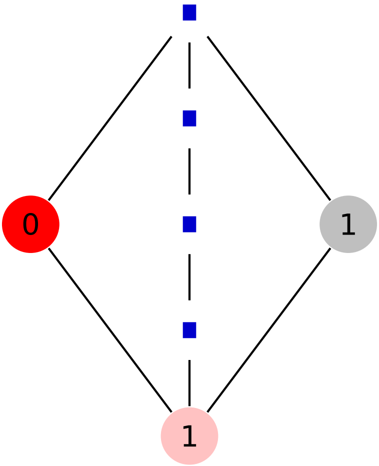

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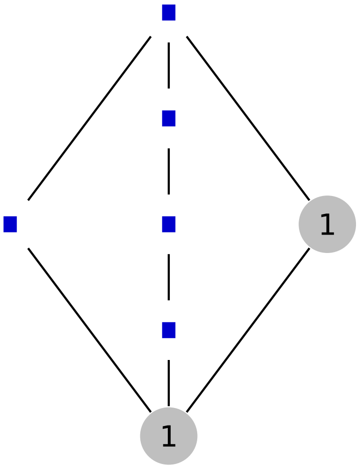

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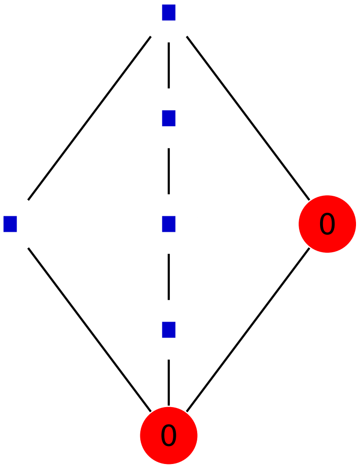

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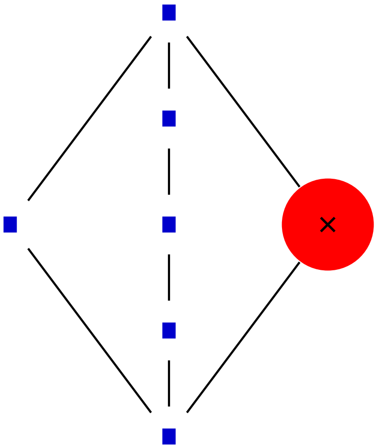

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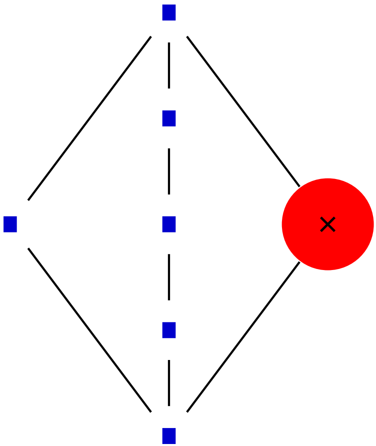

- Lister wins on $\Theta_{2,2,4}$ when each vertex has 2 tokens.

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Ex. $\mathrm{sp}(\Theta_{2,2,4}) = 2(4+3)+1 > \mathrm{sc}(\Theta_{2,2,4})$.

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Thm. The following graphs are sc-greedy:
(Isaak [2004]) Graphs whose blocks are sc-greedy
(BBBD [2006]) K_n, C_n, trees, line graphs of trees

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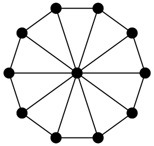

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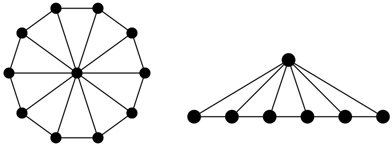

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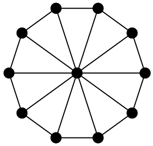 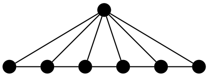 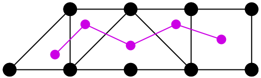

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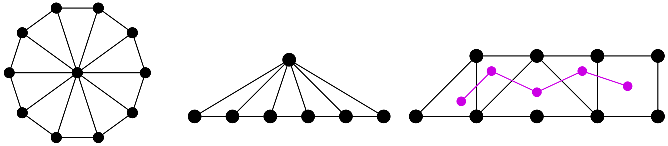

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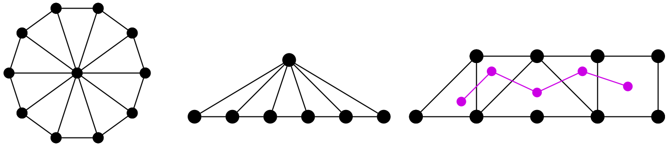

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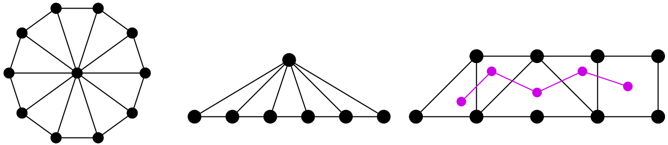

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Def. G is weakly sp-greedy when $G - v$ is sp-greedy $\forall v$.

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In general, neither class is sc-greedy.

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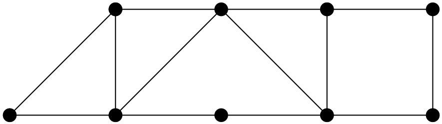
Let G be pathic and f-paintable where $\sum f(v) = (G)$.



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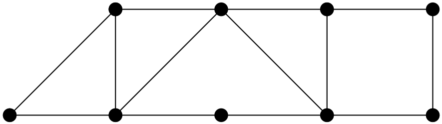

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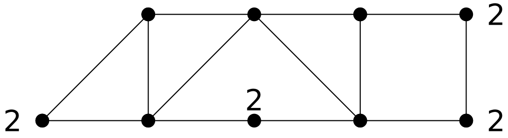

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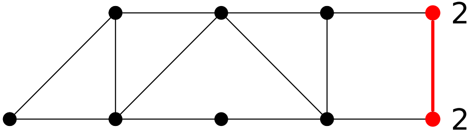

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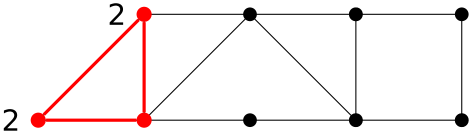

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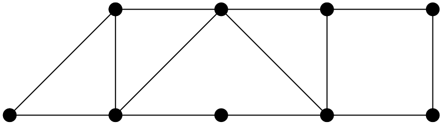

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Therefore G sp-greedy.

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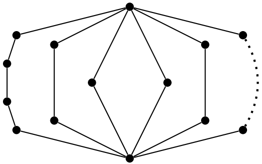
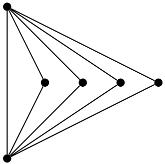
Def. The generalized theta-graph $\Theta_{k_1,\ldots,k_r}$ consists of two vertices joined by internally disjoint paths of lengths $k_1 \leq \cdots \leq k_r$.





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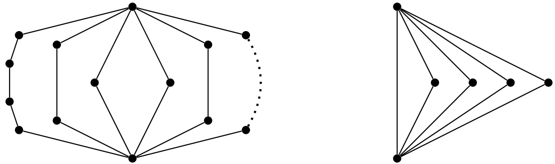

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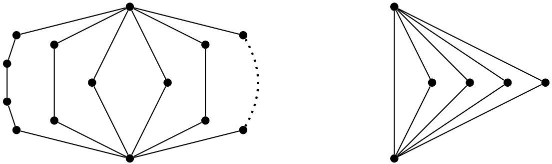

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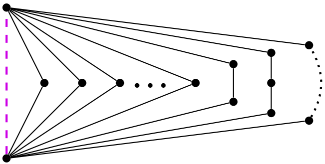

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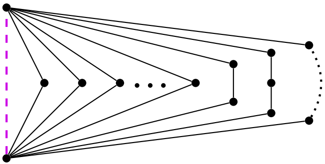

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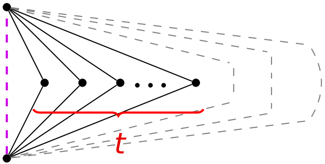

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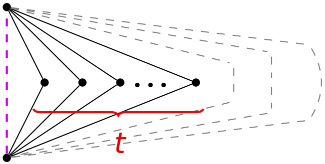

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- Knowing $\mathrm{sp}(K_{2,t})$ and $\mathrm{sp}(B_t)$ completes the result.

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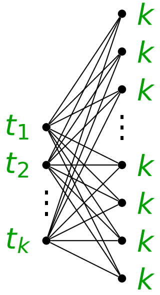

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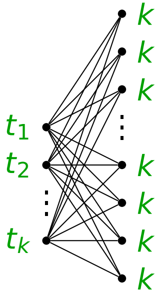

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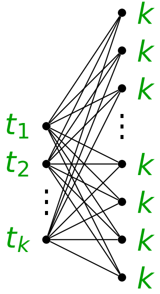

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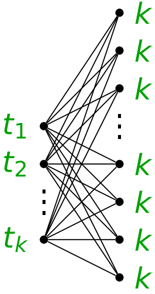

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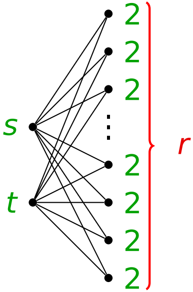

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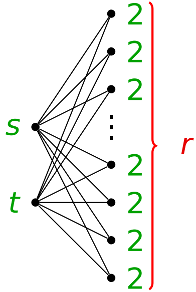

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Upper bound: Use previous theorem:

- Painter wins in $K_{k,r} \iff r < \prod_{i=1}^{k} t_i$
- s and t tokens on the left vertices.
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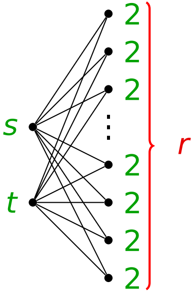

Proof for $\mathrm{sp}(K_{2,r})$

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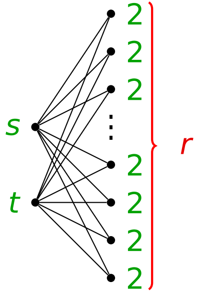
Painter wins, proving the upper bound.



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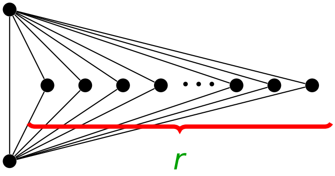

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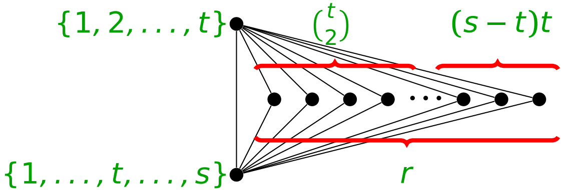

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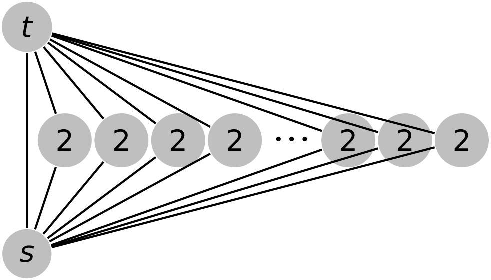

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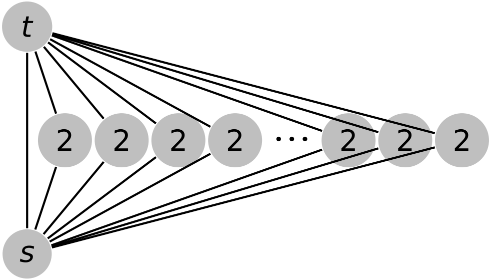

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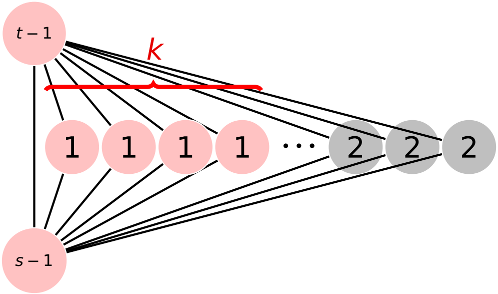

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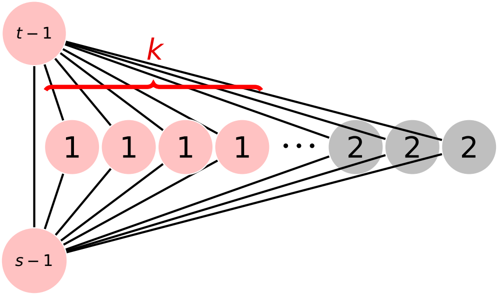

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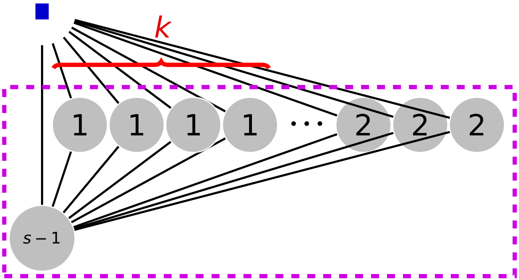

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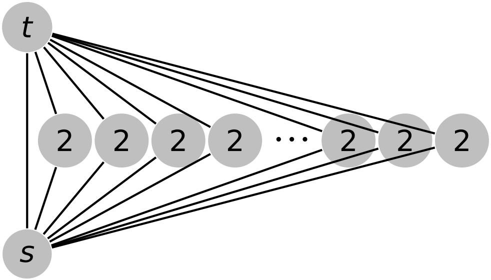

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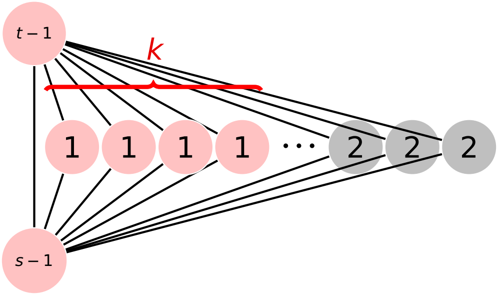

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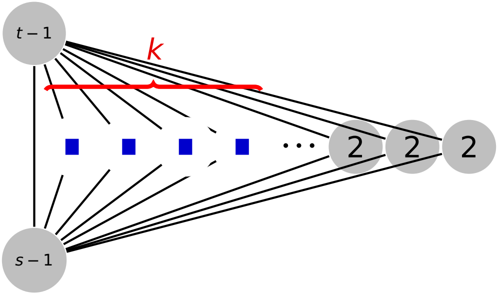

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Case 2: $k \geq s - 1$. Painter wins in B_{r-k} by induction.

Other Results and Questions

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Thank you for coming!