Online List Coloring of Graphs

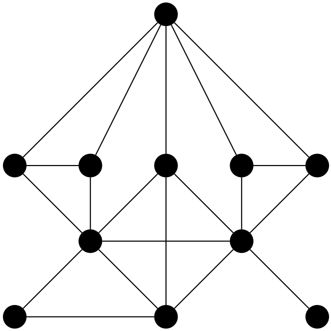
Thomas Mahoney

University of Illinois at Urbana-Champaign
tmahone2@math.uiuc.edu

January 28, 2015

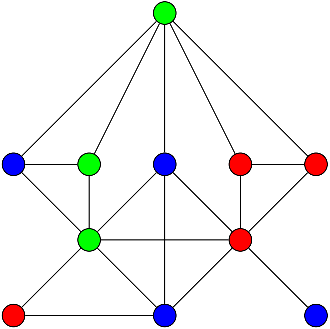
Graph Coloring

We use a graph to model objects (vertices) and relationships (edges) where each edge indicates a relationship between a pair of vertices.



Graph Coloring

Def. A vertex coloring is an assignment of colors or labels to the vertices of a graph.



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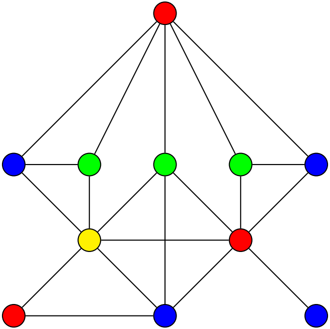

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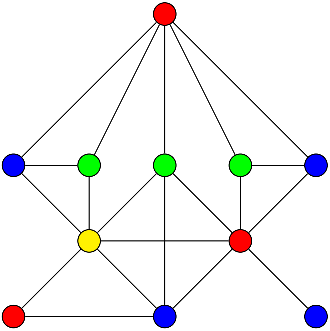
Appl. (Scheduling) Edges represent time conflicts; a proper coloring is a schedule that is conflict-free.



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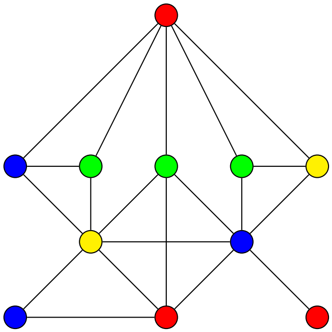
Def. An r-dynamic vertex coloring is a proper coloring in which each vertex v has at least $\min\{r, d(v)\}$ distinct colors in its neighborhood.



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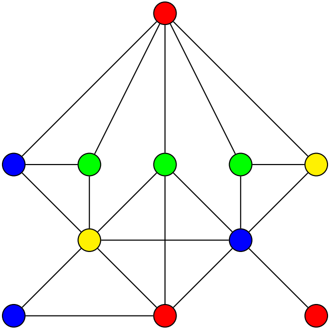
Appl. (Information Sharing) Each color is a packet of information; r-dynamic coloring requires each vertex v to have access to $\min\{r, d(v)\}$ distinct packets.



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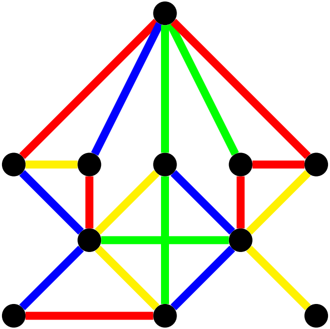
Def. An edge coloring is an assignment of colors or labels to the edges of a graph.



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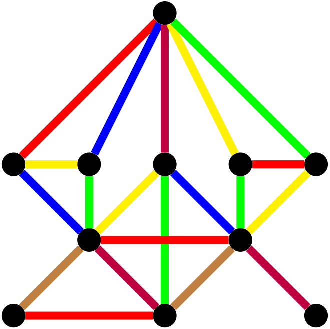
Def. An edge coloring is proper when edges sharing a common endpoint have distinct colors.



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Ques. What if each vertex has a list of available colors?

List Coloring (Graph Choosability)

Originators: Vizing [1976], Erdős–Rubin–Taylor [1979]:

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- Since all lists may be the same, always $\mathrm{ch}(G) \geq \chi(G)$.

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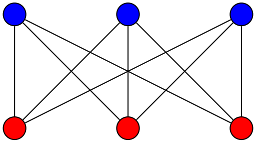

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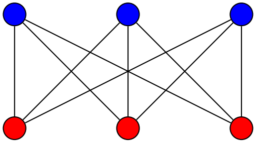

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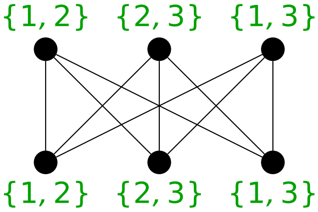

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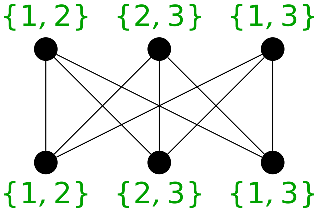

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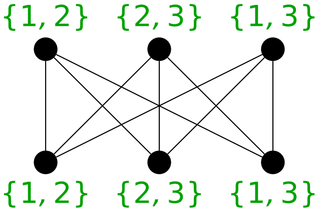

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Worst-case analysis is modeled by the following game:

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Two players: Lister and Painter on a graph *G* with a positive number of tokens at each vertex.

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Appl. (Mahoney)

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Appl. (Mahoney) Each day Wife names rooms to paint. Husband won't color adjacent rooms the same. If Wife names a room more than twice, there is trouble.

Can Husband paint the house without trouble?

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• An adaptive Lister, responding to Painter's earlier moves, may do better.

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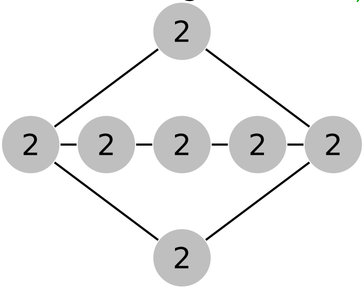

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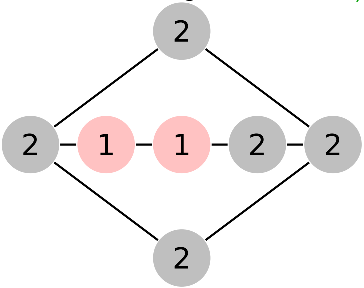

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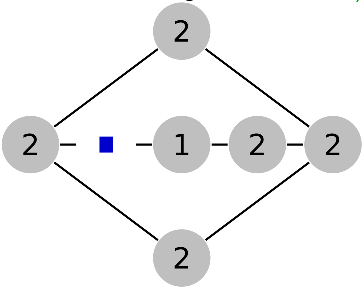

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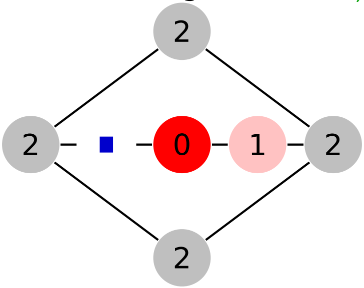

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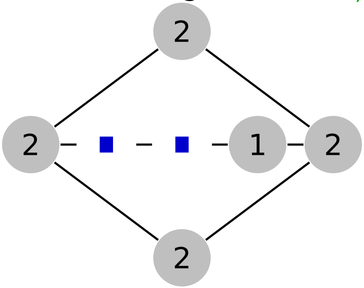

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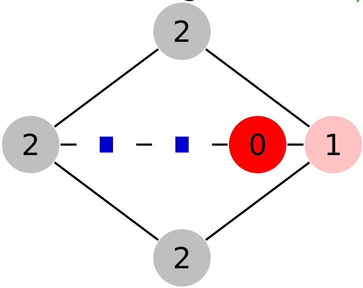

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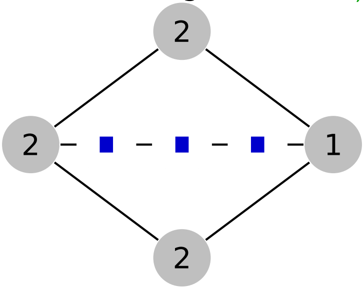

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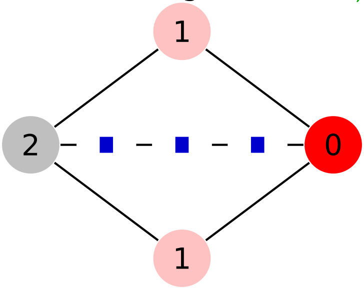

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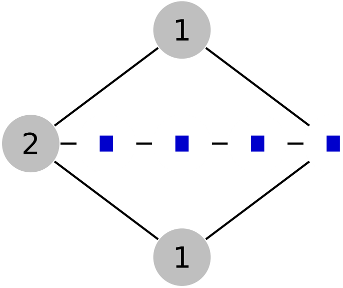

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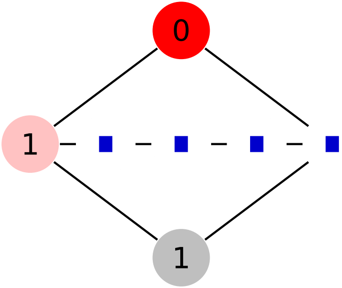

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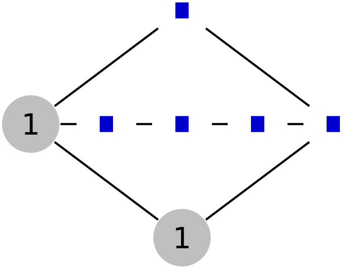

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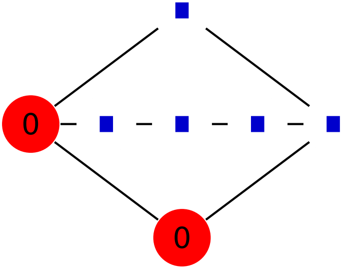

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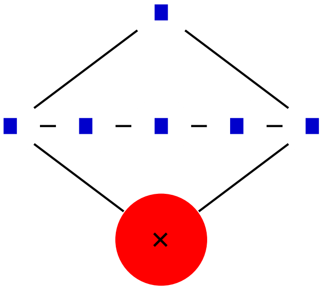

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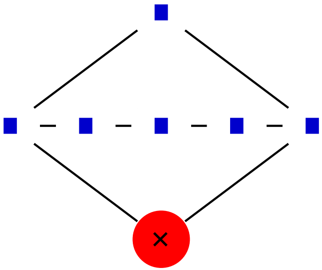

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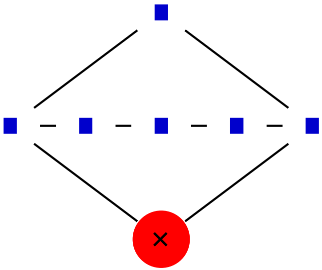

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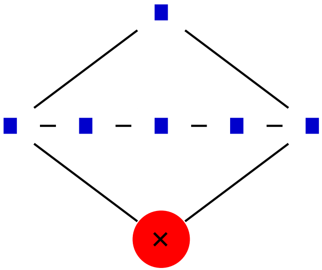

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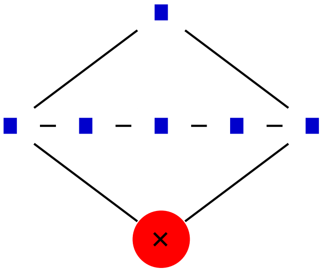

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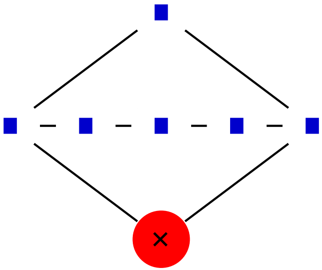

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Ques. (Annoying!) Is $\mathring{\text{ch}}(G) - \text{ch}(G) > 1$ for some G?

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Tournament Scheduling (Schauz [2010])

The complete graph K_k is

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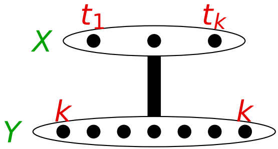

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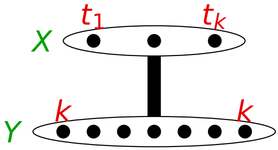

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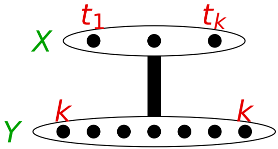

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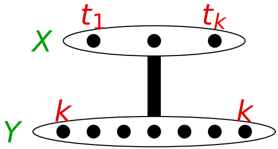

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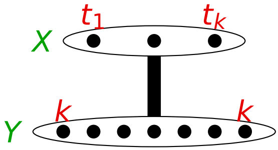

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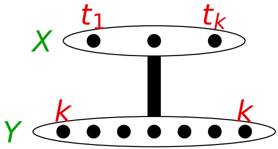

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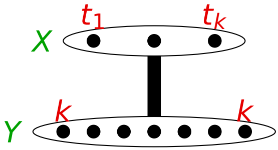

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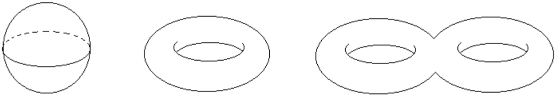
Def. The genus of a graph G, denoted $\gamma(G)$, is the min genus of an orientable surface on which G embeds.



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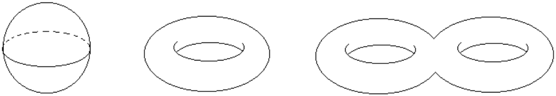

Def. The r-dynamic chromatic number, denoted $\chi_r(G)$, is the least k such that G is r-dynamically k-colorable.

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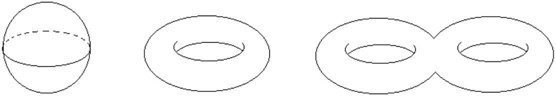

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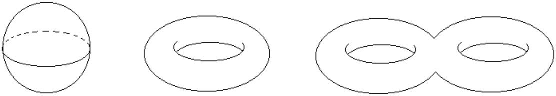

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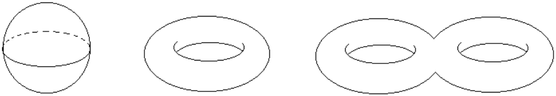

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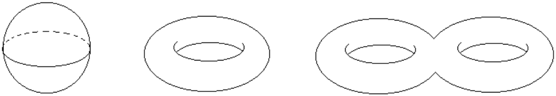

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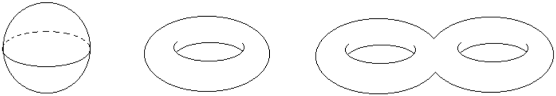

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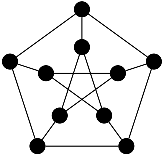

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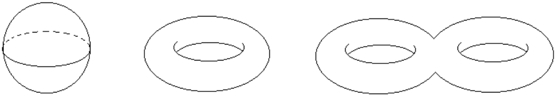

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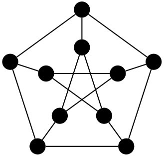

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Thm. For any genus γ and any $r \ge 1$, we prove an upper bound on $\mathring{\mathrm{c}}\mathrm{h}_r(G)$ when $\gamma(G) \le \gamma$.

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Lem. When $r \geq 15$, the following configurations are reducible for r-dynamic $(7r + 10)$-paintability:

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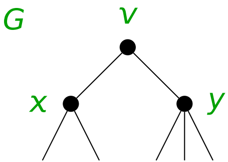

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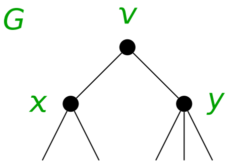

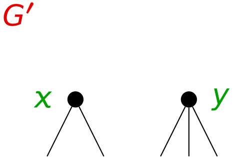

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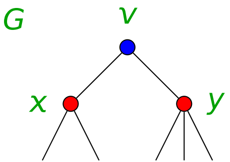
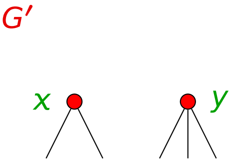
No color for v produces a 3-dynamic coloring!





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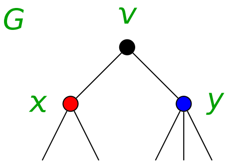

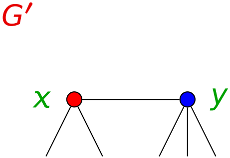

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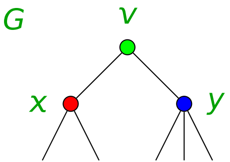
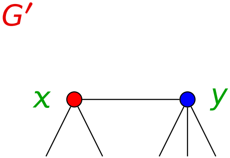
Painter skips coloring v at most $(r-1)d(v) + d(v)$ times, and $7r+10$ tokens are available at v. ■





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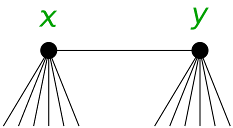

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Thm. Let G be a graph, and let $\gamma = \gamma(G)$.

$\gamma \le 2$ and $r \ge 2\gamma + 11 \Rightarrow \mathring{\check{c}}\mathrm{h}_r(G) \le (\gamma + 5)(r + 1) + 3$

$\gamma \ge 3$ and $r \ge 4\gamma + 7 \Rightarrow \mathring{\check{c}}\mathrm{h}_r(G) \le (2\gamma + 2)(r + 1) + 3$

$\gamma \le 2$ and $r < 2\gamma + 11 \Rightarrow \mathring{\check{c}}\mathrm{h}_r(G) \le (\gamma + 5)(2\gamma + 12) + 3$

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- Natural Inequality: $\mathring{\mathrm{sch}}(G) \geq \mathrm{sch}(G)$.

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Def. A graph G is sp-greedy when $\mathrm{s\mathring{c}h}(G) = \sigma(G)$
(and is sc-greedy when $\mathrm{sch}(G) = \sigma(G)$).

Obs. sc-greedy $\Rightarrow$ sp-greedy. ($\mathrm{sch}(G) \le \mathrm{s\mathring{c}h}(G) \le \sigma(G)$)

Thm. The following graphs are sc-greedy:
(Isaak [2004]) Graphs whose blocks are sc-greedy
(BBBD [2006]) K_n, C_n, trees, line graphs of trees

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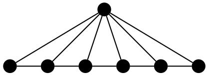

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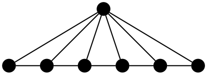

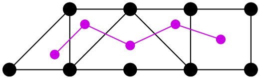

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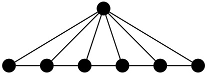

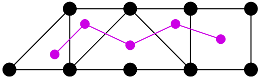

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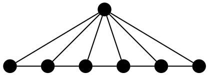
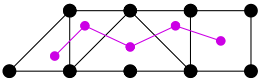

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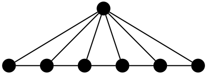

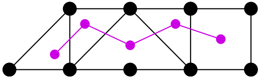

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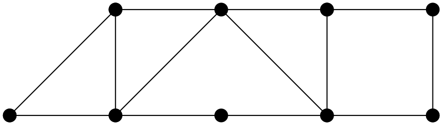
Lem. Suppose G is weakly sp-greedy, f-paintable, and $\sum f(v) = \mathring{\mathrm{sch}}(G)$; G is sp-greedy if

- D forms a clique and $f(u) + |D| - 1 \ge d_G(u) + 1$ $\forall u \in D$,
- $f(v) = 1$ for some v, or
- $\{u, v, w\}$ forms $\triangle$, $d(u) \le d(v) = 3$, and $f(u) = f(v) = 2$.

Pathic graphs are sp-greedy

Let G be pathic and f-paintable where $\sum f(v) = \mathring{\text{sch}}(G)$.



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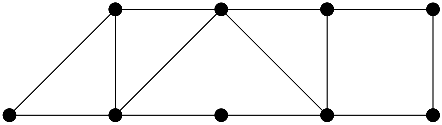

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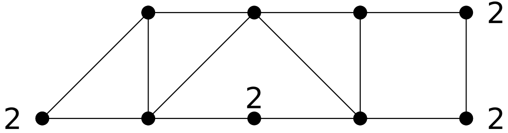

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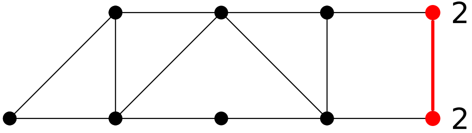

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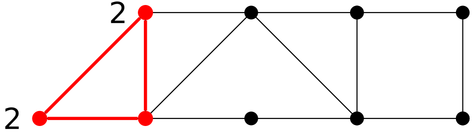

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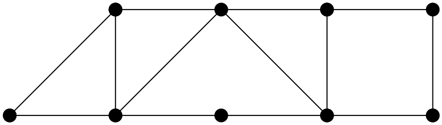

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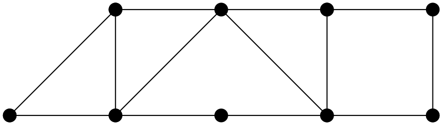

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- (MTW [2014]) More partial results towards Conjecture.

Conclusions and Open Questions

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Goal: Better bounds on $\mathring{\text{ch}}(K_{n,n})$.

Conj. (Zhu [2009]) $|V(G)| \leq 2\chi(G) \;\Rightarrow\; \chi(G) = \mathring{\text{ch}}(G)$.

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Thank You!