Extending graph choosability results to paintability

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Joint work with
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Gregory J. Puleo, Mu-Tsun Tsai, and Douglas West

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Appl.

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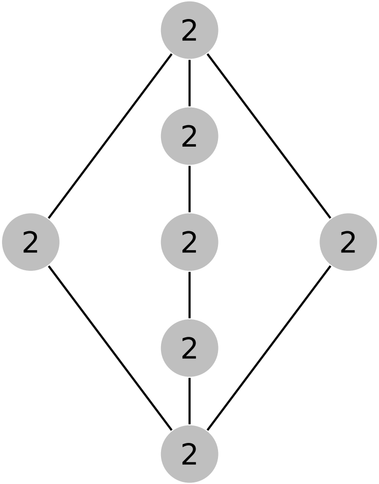
Appl. Wife doesn't want to tell me more than 2 times to paint any one room. Can I stay out of trouble?

Example Game

Let's play the Marker/Remover game on $\Theta_{2,2,4}$.



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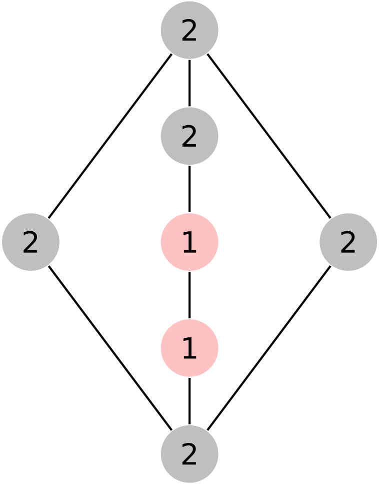

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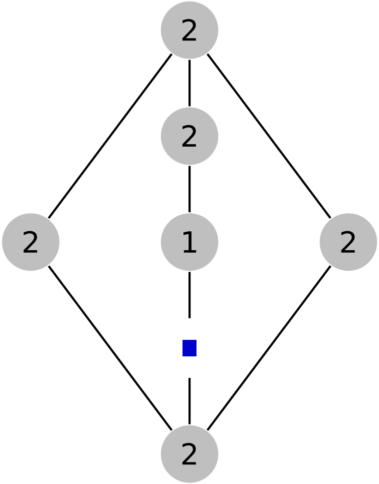

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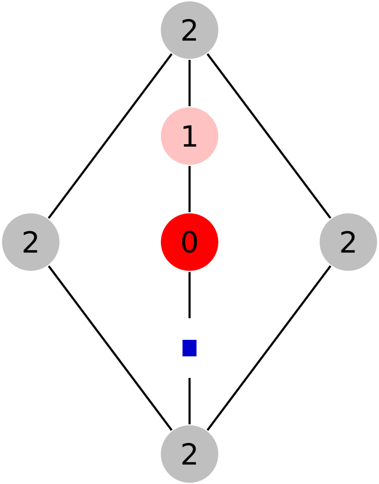

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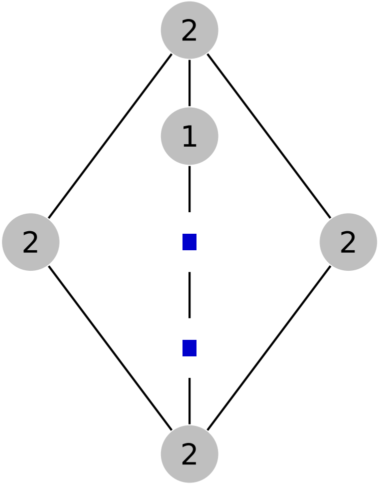

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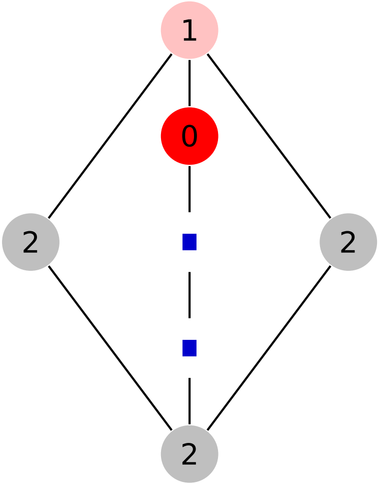

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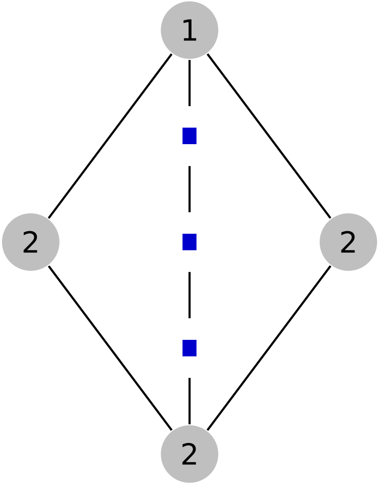

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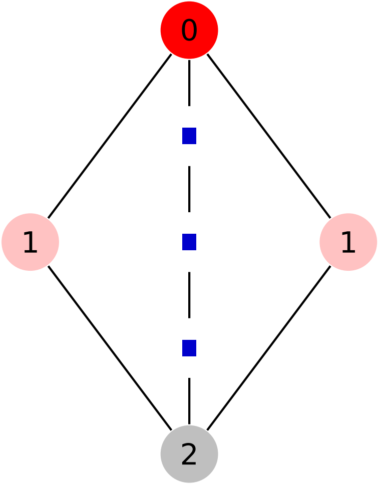

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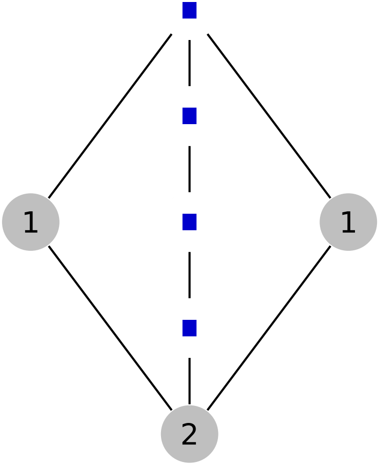

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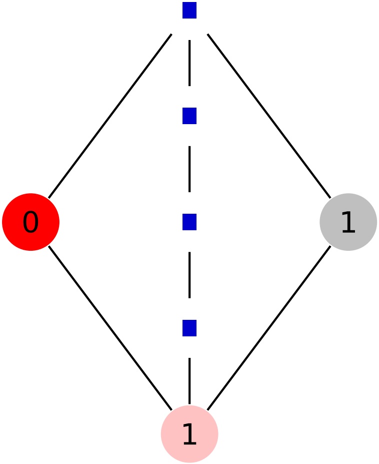

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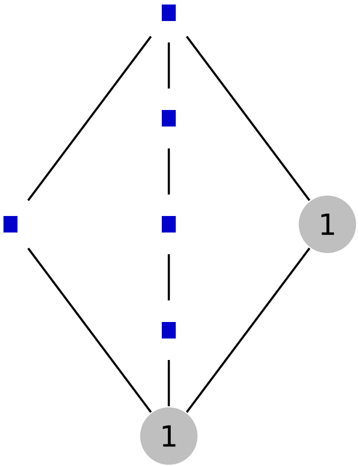

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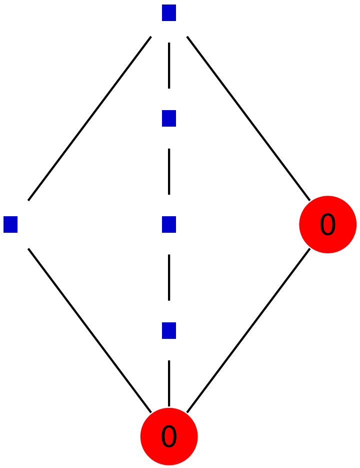

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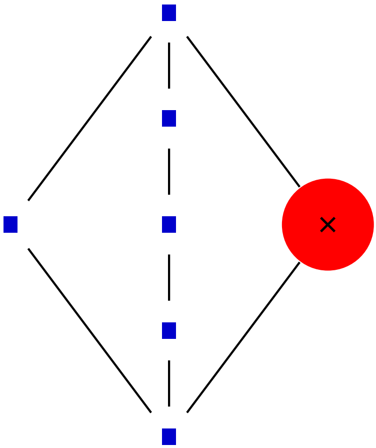

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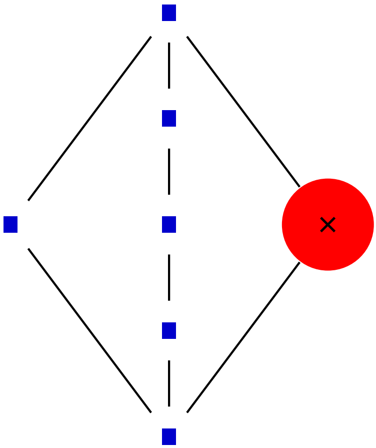

Conclude: Marker has a winning strategy on this graph when each vertex has 2 tokens.

Definitions

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Def. If $f(v) = k$ for all $v \in V(G)$ and G is f-paintable, then we say G is k-paintable.

Def. The least such k for which this is true is the paintability or paint number of G and is denoted $\chi_p(G)$.

Relation to Chromatic Number

Obs. Sets removed by Remover form a proper coloring.

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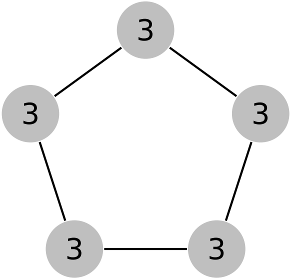
Prop. $\chi(C_5) = 3$.

Ex. Consider the following strategy for Marker:



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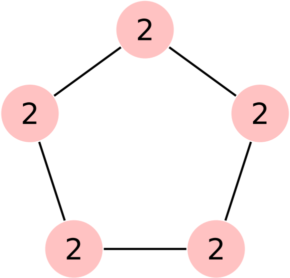

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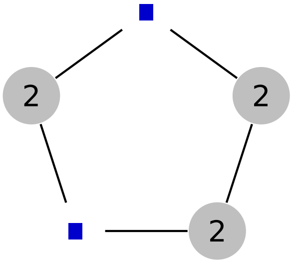

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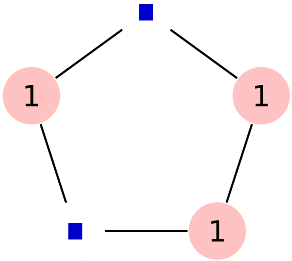

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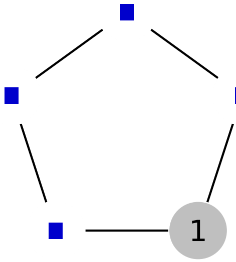

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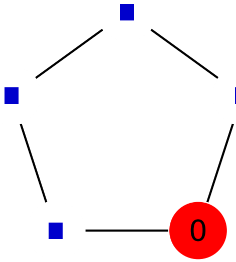

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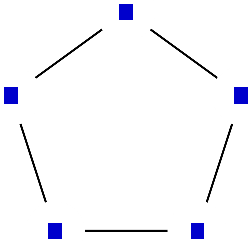

Obs. If Marker always marks all available vertices, then the least k such that Remover can win against this strategy is $\chi(G)$.

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Def. The least such k for which this is true is the choosability of G and is denoted $\chi_\ell(G)$.

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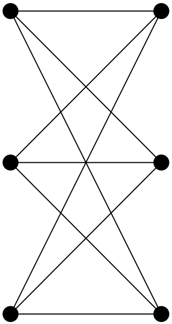

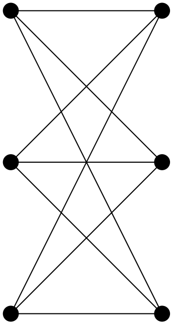

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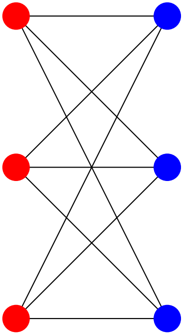

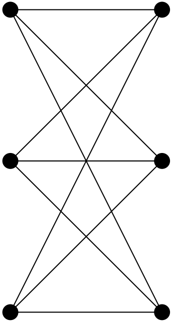

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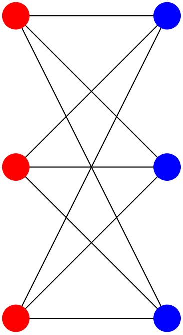

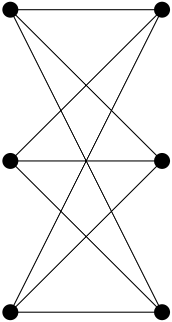

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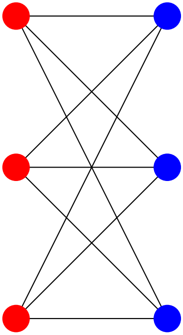

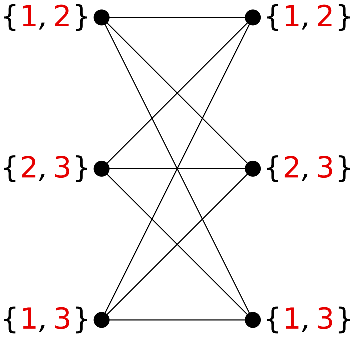

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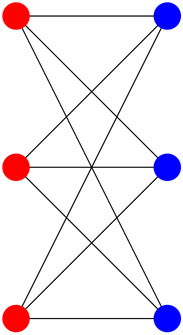

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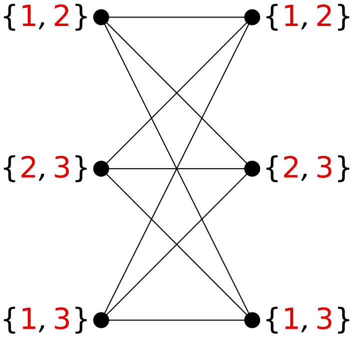

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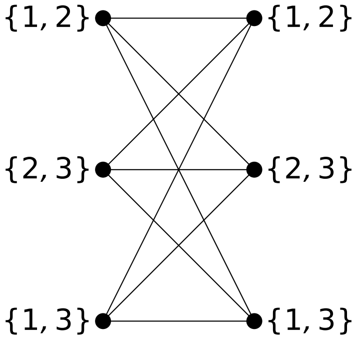

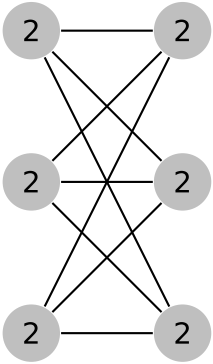

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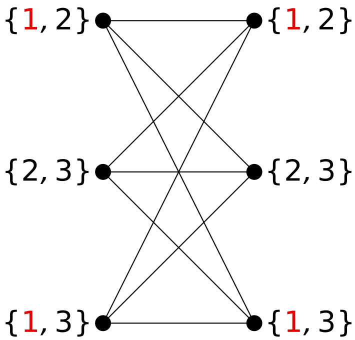

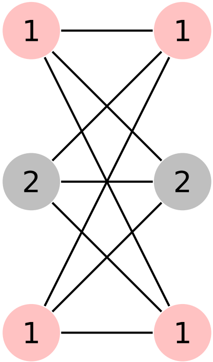

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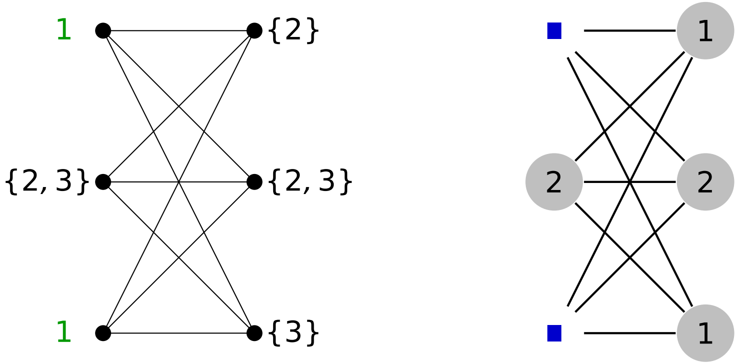

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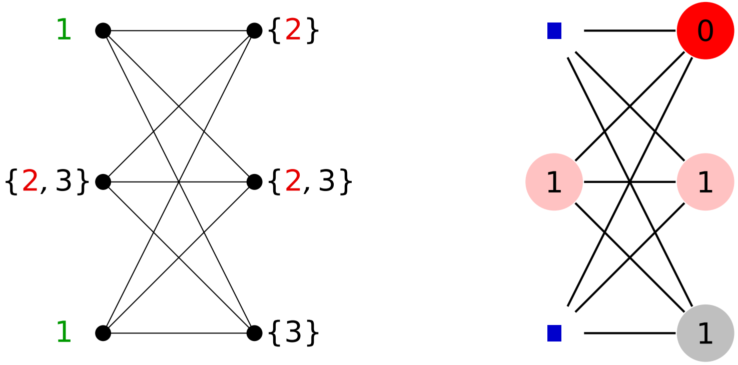

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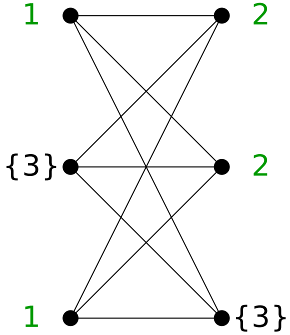

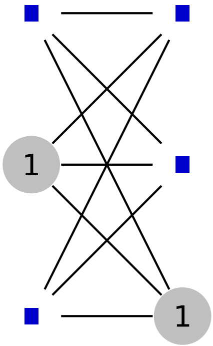

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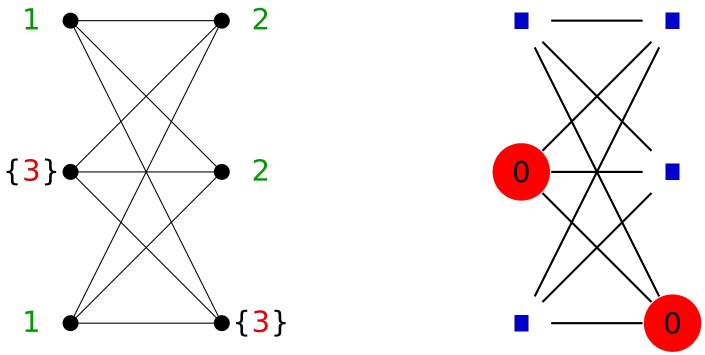

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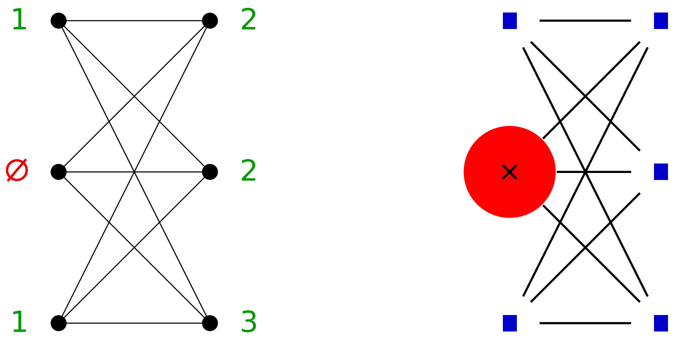

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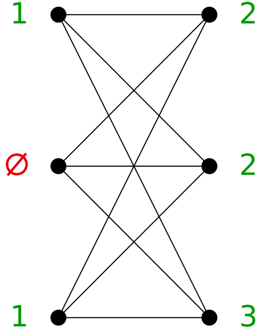
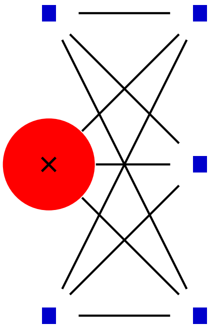
Obs. If Marker's strategy mimics list assignments by marking vertices whose list has color i on the ith round, then the least k such that Remover has a winning strategy against all L having list of size k is $\chi_\ell(G)$.





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Obs. Similarly, Marker could list moves ahead of time, but an adaptive strategy may be better.

Background

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Some coloring upper bounds also hold for list coloring, strengthening the result. Similarly, some list coloring upper bounds hold for paintability.

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Ques. What analogous bounds hold for paintability?

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Thm. For any graph G, there exists $t_0 \in \mathbb{N}$ such that if $t > t_0$, then $G \oplus K_t$ is chromatic-paintable.

Claw-free Perfect Graphs

Def. A graph G is claw-free if it contains no induced copy of $K_{1,3}$.

Def. Given a graph G, the clique number of G, denoted $\omega(G)$, is the size of the largest clique.

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Using similar techniques, we strengthened this result:

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Thm. A graph is 3-paint-critical if and only if it is

1. an odd cycle
2. two edge disjoint even cycles joined by one path
3. $\Theta_{2r+1,2s+1,2t+1}$ with $r > 0, s > 0, t \geq 0$
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Obs. The characterizations are the same except in classes 4 and 5 due to $\chi_\ell(\Theta_{2,2,2n}) < \chi_p(\Theta_{2,2,2n})$.

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Obs. Determining when $K_{\ell,r}$ is $(\ell - 1)$-paintable is different and more complicated than $(\ell - 1)$-choosable.

Sum-Choosability and Sum-Paintability

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Obs. The natural inequality $\chi_{sc}(G) \le \chi_{sp}(G)$ holds.

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Cor. These Lemmas determine the sum-paintability of generalized theta graphs.

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Obs. Adding leaves and ears of length at least 3 to an sp-greedy graph creates another sp-greedy graph.

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Obs. Thus the question for sp-greediness of outerplanar and chordal graphs remains open.

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Ques. What other choosability results hold for paintability?