Online sum list coloring of graphs

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Joint work with
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Def. (Isaak [2002]) The sum-choosability $sc(G)$ is the least $\sum_{v \in V(G)} f(v)$ among f such that G is f-choosable.

Online List Coloring (Zhu [2009])

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Worst-case analysis is modeled by the following game:

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Two players: Lister and Painter on a graph G with a positive number of tokens at each vertex.

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- Painter is creating a proper coloring of G.

Example Game

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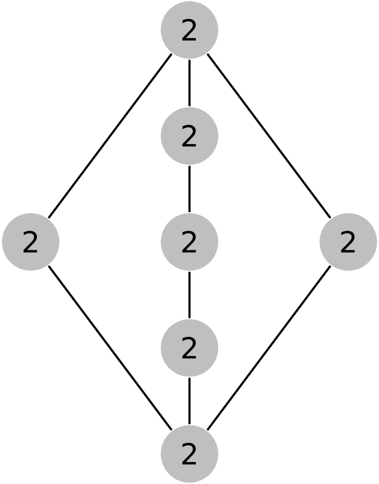

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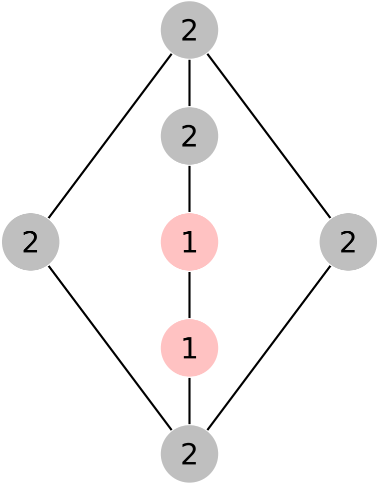

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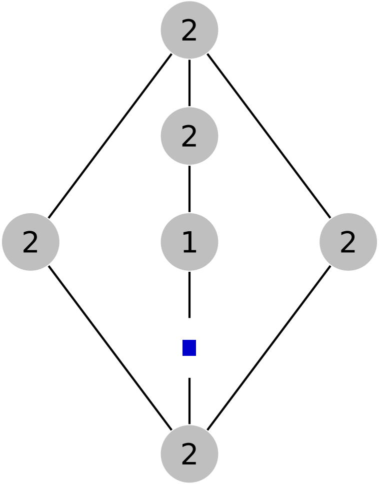

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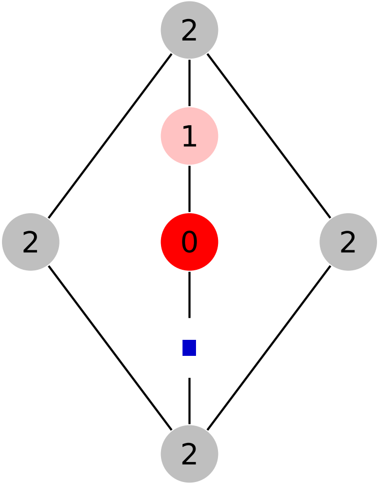

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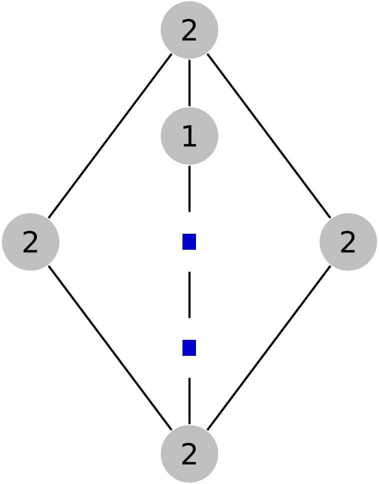

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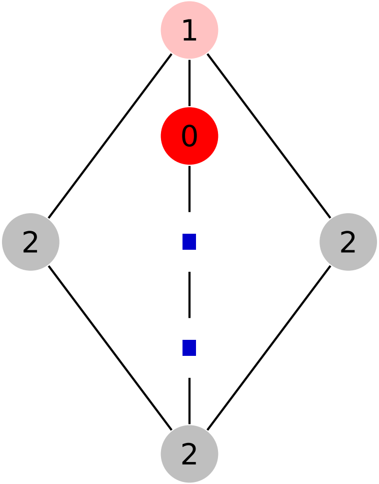

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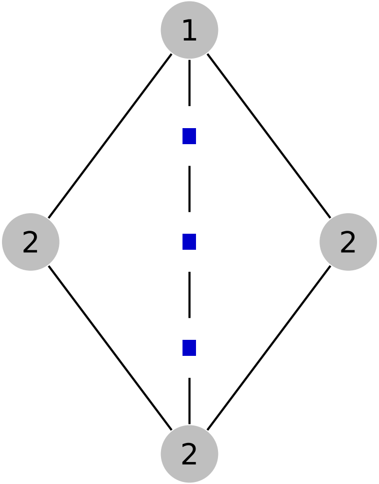

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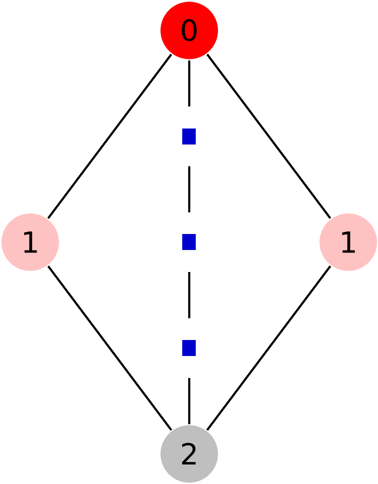

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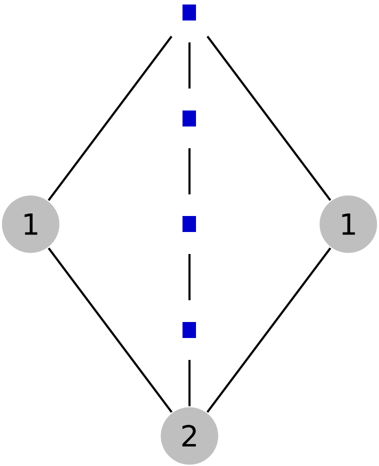

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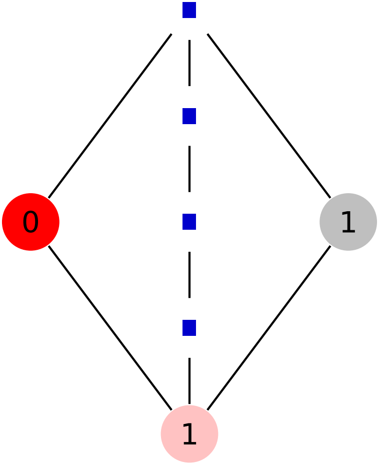

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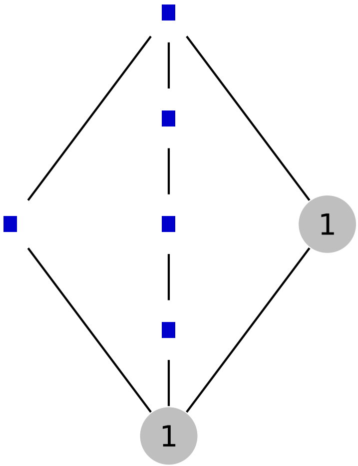

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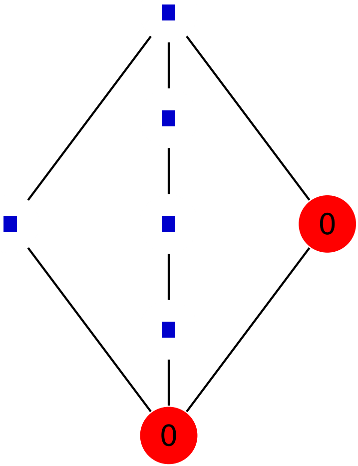

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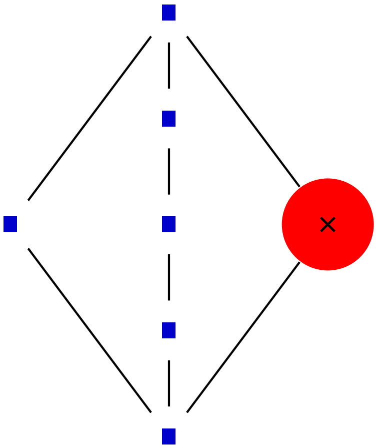

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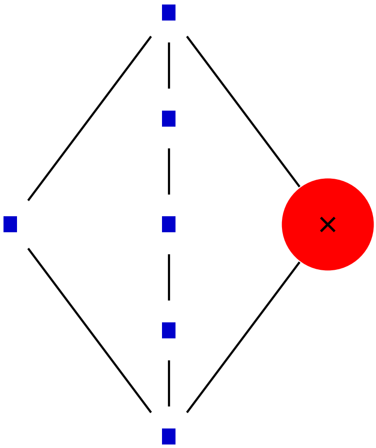

- Lister wins on $\Theta_{2,2,4}$ when each vertex has 2 tokens.

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Ex. $\mathrm{sp}(\Theta_{2,2,4}) = 2(2r+3)+1 > \mathrm{sc}(\Theta_{2,2,4})$.

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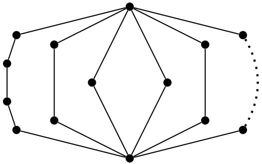

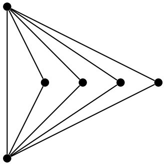

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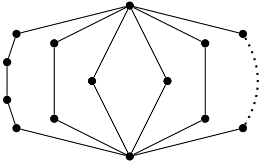

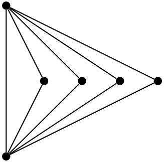

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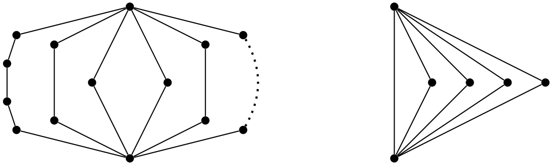

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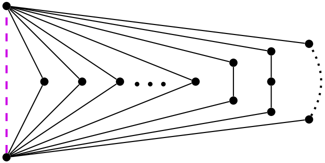
Thm. $\mathrm{sp}(B_r) = 2r + \min\{s + t : t(s - t) + \binom{t}{2} > r\}$.

Generalized Theta-Graphs (Cont.)

Thm. If G is the generalized theta-graph $\Theta_{k_1,\ldots,k_r}$, then

$$\mathrm{sp}(G) = \begin{cases} |V(G)| + |E(G)|, & \text{if } \ell_3 > 2 \\ \mathrm{sp}(K_{2,t}) + \sum_{i=t+1}^{k} (2\ell_i - 1), & \text{if } \ell_1 = \cdots = \ell_t = 2 \\ \mathrm{sp}(B_{t-1}) + \sum_{i=t+1}^{k} (2\ell_i - 1), & \text{if } \ell_1 = 1, \ell_2 = \cdots = \ell_t = 2 \end{cases}$$



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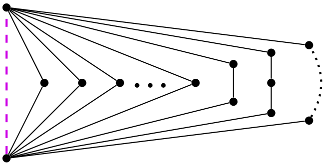

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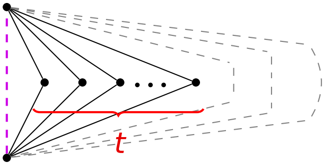

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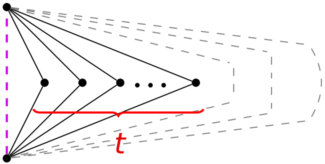

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- Knowing $\mathrm{sp}(K_{2,t})$ and $\mathrm{sp}(B_t)$ completes the result.

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If $f(y) = k$ for $y \in Y$ and $f(x_i) = t_i$ for $x_i \in X$, then
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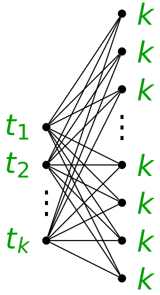

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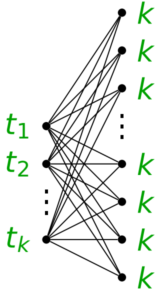

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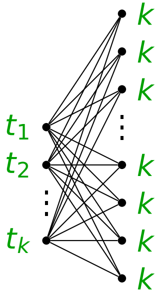

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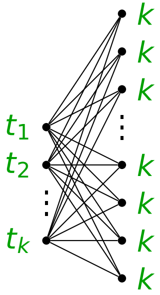

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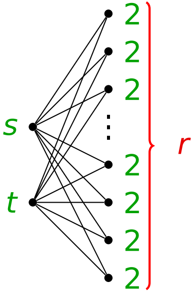

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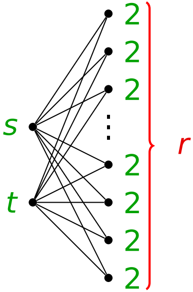

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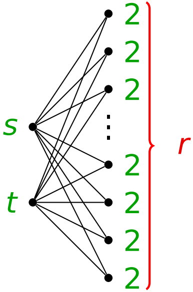

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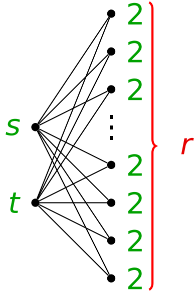

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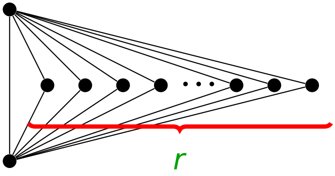

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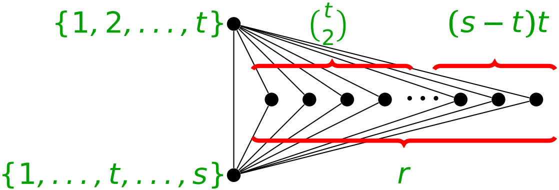

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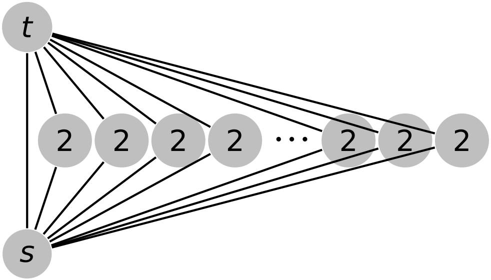

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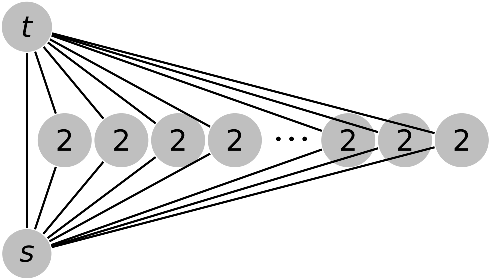

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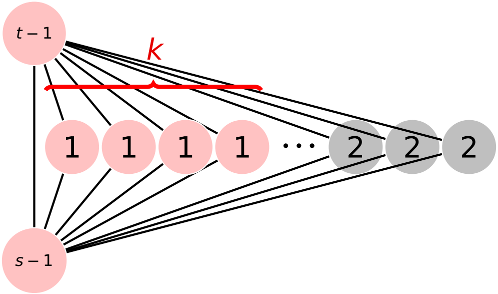

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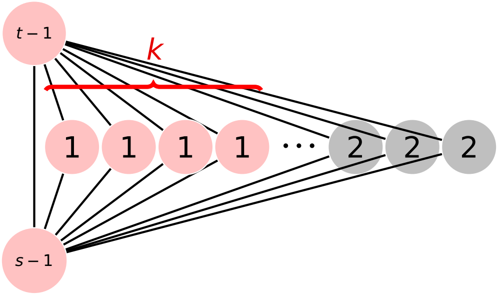

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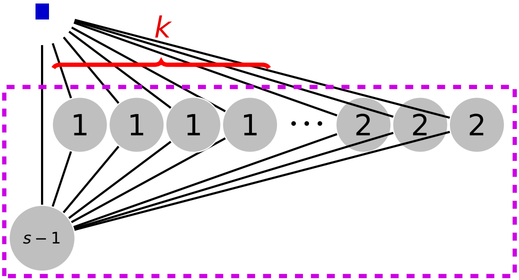

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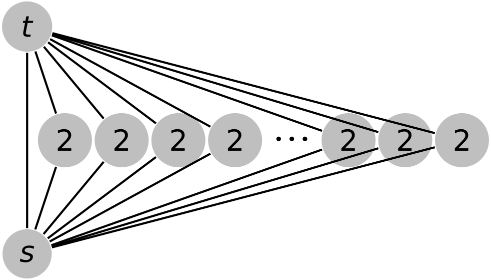

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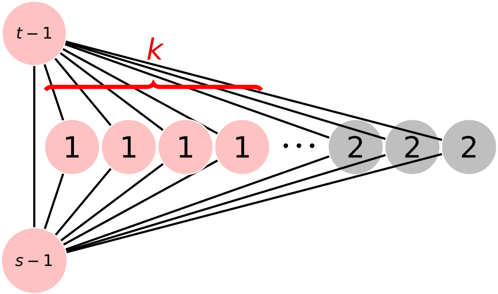

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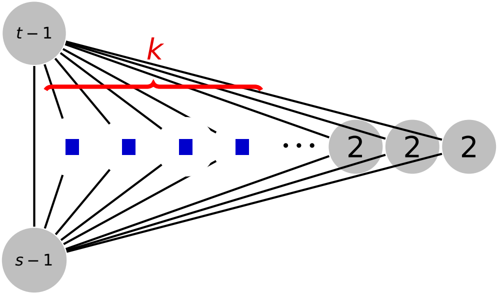

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Thm. The following graphs are sc-greedy:
(Isaak [2004]) Graphs whose blocks are sc-greedy
(BBBD [2006]) K_n, C_n, trees, line graphs of trees

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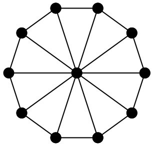

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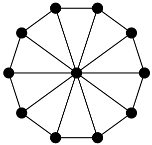

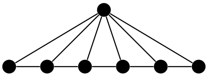

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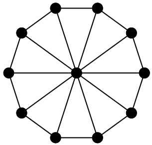

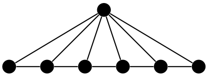

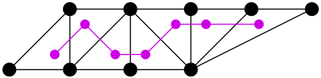

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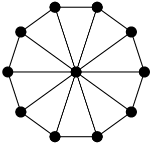
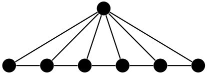
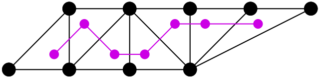

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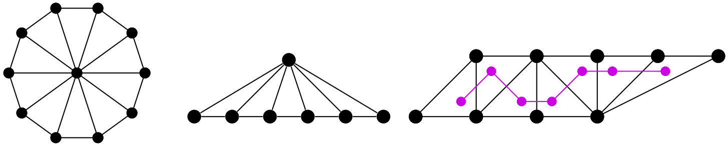

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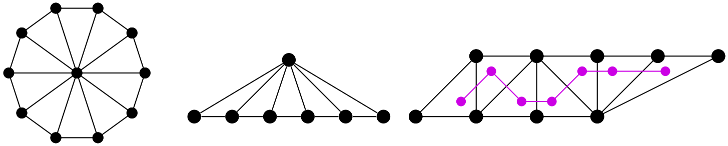

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Def. We say G is weakly sp-greedy when $G - v$ is sp-greedy for any $v \in V(G)$.

Main Tool and Result

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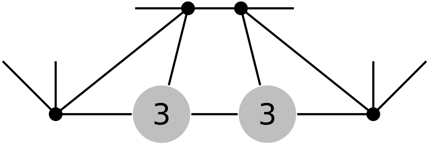

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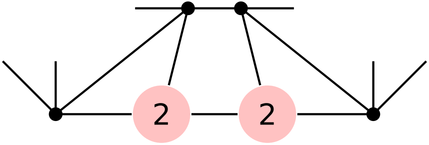

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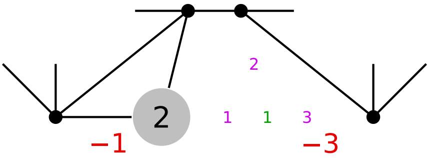

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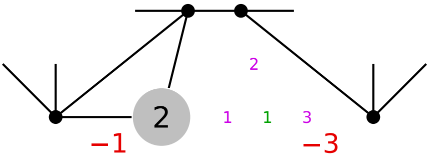

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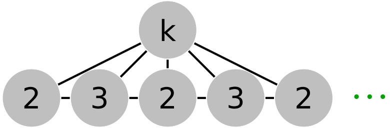

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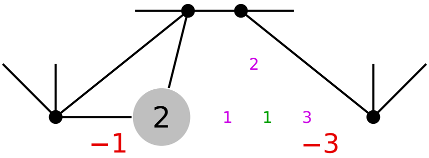

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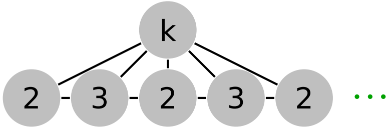

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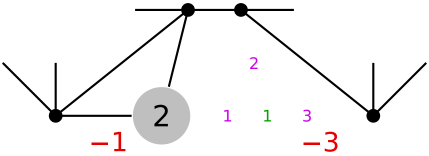

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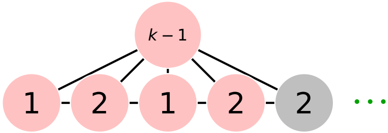

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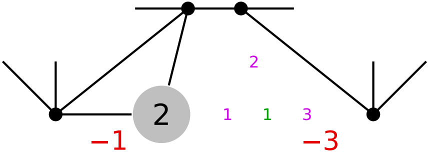

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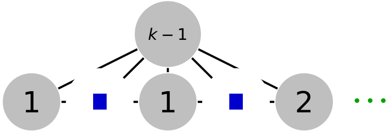

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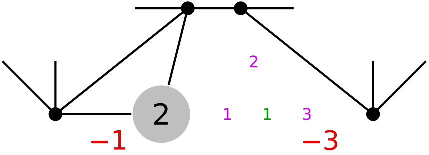

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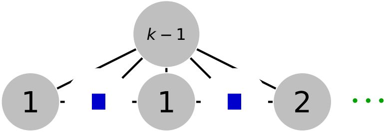

Marking each leaf and its neighbor we have F_{n-4} with 8 edges, 4 vxs, and 13 tokens lost.

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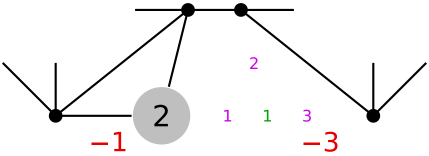

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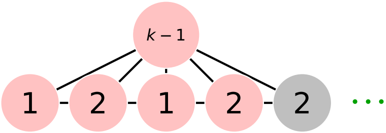

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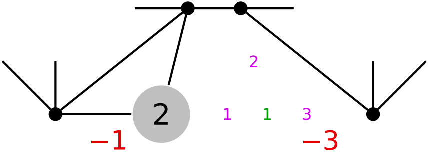

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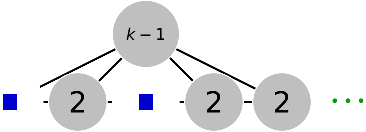

Marking each leaf and its neighbor we have F_{n-3} with 6 edges, 3 vxs, and 9 tokens lost.

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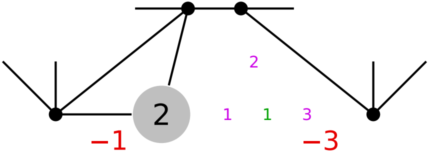

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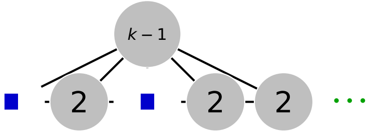

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By Induction, F_{n-3} and F_{n-4} are weakly sp-greedy, so our Tool implies the result.

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• Adding an ear of length ℓ does not necessarily increase $\mathrm{sc}(G)$ by $2\ell - 1$. (Consider $\Theta_{2,2,4}$)

End