Characterization of $(2m, m)$-paintable graphs

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Joint work with
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List Coloring \(((a, b)\)-choosability)\]

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Results in \((a, b)\)-choosability

**Thm.** (Alon 1993) Given \((a, b, n)\), \(\exists m > 1\) such that every \((a, b)\)-choosable graph on \(n\) vxs is \((am, bm)\)-choosable.
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**Thm.** (Meng–Puleo–Zhu 2014+) Characterized the 3-choice-critical graphs that are \((4, 2)\)-choosable.
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**Online Version:** Coloring algorithm can’t see entire lists. **Worse-case** analysis modeled by the following game:
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- Adjacent color sets are disjoint \(\Rightarrow\) Proper \((L, g)\)-coloring
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**Def.** For $f : V(G) \rightarrow \mathbb{N}$ and $g : V(G) \rightarrow \mathbb{N}$, we say $G$ is $(f, g)$-paintable if Painter has a winning strategy in the Lister/Painter game when each vertex $v$ starts with $f(v)$ tokens and needs to receive $g(v)$ colors.
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**Thm.** (Gutowski 2011) $\inf$ cannot be replaced by $\min$!
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**Thm.** The \( \min t \) s.t. \( C_{2k+1} \) is \((t,m)\)-paintable is \( 2m + \lceil m/k \rceil \).
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(≤) Let \( V(G) = \{ \nu_0, \ldots, \nu_{2k} \} \), and orient cyclically.
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If Lister wins when $t = 2m + \lceil m/k \rceil$, then on some round $\exists v_i$ that has been “rejected” $m + \lceil m/k \rceil + 1$ times.
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If Lister wins when \(t = 2m + \lceil m/k \rceil\), then on some round \(\exists v_i\) that has been “rejected” \(m + \lceil m/k \rceil + 1\) times. Each time, \(v_{i-1}\) was marked, but has received \(\leq m\) colors. So \(v_{i-1}, v_i\) are marked and not colored \(\geq \lceil m/k \rceil + 1\) times.
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On the \( i \)th round of **Lister** marking all \( 2k + 1 \) vxs, **Painter** colors \( \{v_i, v_{i+2}, \ldots, v_{i+2k-2}\} \) (mod \( 2k + 1 \)).

If **Lister** wins when \( t = 2m + \lceil m/k \rceil \), then on some round \( \exists v_i \) that has been “rejected” \( m + \lceil m/k \rceil + 1 \) times. Each time, \( v_{i-1} \) was marked, but has received \( \leq m \) colors. So \( v_{i-1}, v_i \) are marked and not colored \( \geq \lceil m/k \rceil + 1 \) times.

Only happens once every \( 2k + 1 \) rounds. \( \Rightarrow \) All vxs lost \( \lceil m/k \rceil (2k + 1) + 1 > t \) tokens.
Foundation for Main Theorem

**Cor.** $C_{2k+1}$ is not $(2m, m)$-paintable for any $m, k \geq 1$. 
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Def. The core of a graph $G$, $\text{core}(G)$, is obtained by iteratively deleting 1-vertices.
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**Thm.** (Zhu 2009) $G$ is $(2, 1)$-paintable $\iff \text{core}(G)$ is
- $K_1$,
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- an even cycle.
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**Thm.** (CLMPTW 2014) $G$ is 3-paint-critical $\iff \text{core}(G)$ is
- an odd cycle,
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- two even cycles joined by a path,
- $\Theta_{r,s,t}$, same parity, not all 2.
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**Cor.** \( C_{2k+1} \) is not \((2m, m)\)-paintable for any \( m, k \geq 1 \).

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**Main Thm.** (MMZ 2014+) Given a graph \( G \) and \( m \geq 1 \), \( G \) is \((2m, m)\)-paintable \(\iff G \) is \((2, 1)\)-paintable.
Proof Outline

(⇒) Every non-(2, 1)-paintable graph has a 3-paint-critical subgraph.
Proof Outline

(⇒) Every non-(2, 1)-paintable graph has a 3-paint-critical subgraph.

• Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4},$, and $\ldots$. 

\begin{center}
\begin{tikzpicture}
\end{tikzpicture}
\end{center}
Proof Outline

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- Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: \(C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}, \), and .
- Using induction on the total number of tokens, give a winning strategy for Lister.
Proof Outline

$(\Rightarrow)$ Every non-$(2, 1)$-paintable graph has a $3$-paint-critical subgraph.

- Reduce the (infinite) family of $3$-paint-critical graphs to $6$ graphs: $C_3$, $K_{2, 4}$, $\Theta_{1, 3, 3}$, $\Theta_{2, 2, 4}$, $\bullet\bullet\bullet\bullet$, and $\bullet\bullet\bullet\bullet$.
- Using induction on the total number of tokens, give a winning strategy for Lister.

$(\Leftarrow)$ Show $K_1$, $C_{2n}$, $K_{2, 3}$ are $(2m, m)$-paintable for $m \geq 1$. 
Proof Outline

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(⇐) Show \(K_1, C_{2n}, K_{2,3}\) are \((2m, m)\)-paintable for \(m \geq 1\).

• \(K_1\) is trivial.
Proof Outline

(⇒) Every non-\((2, 1)\)-paintable graph has a 3-paint-critical subgraph.

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- \(C_{2n}\) is easy.
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Proof Outline

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- Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}, \begin{tikzpicture}[baseline=-.5ex]
    


\end{tikzpicture}$, and $\begin{tikzpicture}[baseline=-.5ex]

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- Using induction on the total number of tokens, give a winning strategy for Lister.

(⇐) Show $K_1, C_{2n}, K_{2,3}$ are $(2m, m)$-paintable for $m \geq 1$.

- $K_1$ is trivial.
- $C_{2n}$ is easy. ($\text{Painter}$ can be greedy)
- $K_{2,3}$ requires more work:
Proof Outline

(⇒) Every non-$(2, 1)$-paintable graph has a 3-paint-critical subgraph.

- Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}, \text{ and } \text{ without a }$.  

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  1) Assign weights to edges.
Proof Outline

$(\Rightarrow)$ Every non-$(2, 1)$-paintable graph has a 3-paint-critical subgraph.

- Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4},$, and $\Theta_{2,4,2}$.

- Using induction on the total number of tokens, give a winning strategy for Lister.

$(\Leftarrow)$ Show $K_1, C_{2n}, K_{2,3}$ are $(2m, m)$-paintable for $m \geq 1$.

- $K_1$ is trivial.

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- $K_{2,3}$ requires more work:
  1) Assign weights to edges.
  2) Prove $\text{Painter}$ can always “balance” the weights.
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⇒) Every non-(2, 1)-paintable graph has a 3-paint-critical subgraph.

• Reduce the (infinite) family of 3-paint-critical graphs to 6 graphs: \( C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}, \), and \( \). 

• Using induction on the total number of tokens, give a winning strategy for Lister.

⇐) Show \( K_1, C_{2n}, K_{2,3} \) are \((2m, m)\)-paintable for \( m \geq 1 \).

• \( K_1 \) is trivial.

• \( C_{2n} \) is easy. (Painter can be greedy)

• \( K_{2,3} \) requires more work:
  1) Assign weights to edges.
  2) Prove Painter can always “balance” the weights.
  3) Show Painter wins if weights remain balanced.
Reducing 3-paint-critical family

**Def.** Given a graph $H$ and $U \subseteq V(H)$, we say $(H, U)$ is an $(a, b)$-gadget if $H$ is $(a, b)$-colorable and every proper coloring gives the same colors to all vertices of $U$. 
Reducing 3-paint-critical family

**Def.** Given a graph $H$ and $U \subseteq V(H)$, we say $(H, U)$ is an $(a, b)$-gadget if $H$ is $(a, b)$-colorable and every proper coloring gives the same colors to all vertices of $U$.

**Ex.** $H = P_{2n+1}, U = \{v_1, v_{2n+1}\}, a = 2m, b = m$. 
Def. Given a graph \( H \) and \( U \subseteq V(H) \), we say \( (H, U) \) is an \((a, b)\)-gadget if \( H \) is \((a, b)\)-colorable and every proper coloring gives the same colors to all vertices of \( U \).

Ex. \( H = P_{2n+1}, U = \{v_1, v_{2n+1}\}, a = 2m, b = m \).

Def. Given a graph \( G \), a vertex \( v \in V(G) \), and an \((a, b)\)-gadget \((H, U)\), an \((H, U)\)-augmentation of \( G \) is obtained from \( G + H \) by splitting \( v \) into \(|U|\) copies, partitioning the edges among those copies, and identifying each copy with a vertex of \( U \).
Reducing 3-paint-critical family

**Def.** Given a graph $H$ and $U \subseteq V(H)$, we say $(H, U)$ is an $(a, b)$-gadget if $H$ is $(a, b)$-colorable and every proper coloring gives the same colors to all vertices of $U$.

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**Def.** Given a graph $G$, a vertex $v \in V(G)$, and an $(a, b)$-gadget $(H, U)$, an $(H, U)$-augmentation of $G$ is obtained from $G + H$ by splitting $v$ into $|U|$ copies, partitioning the edges among those copies, and identifying each copy with a vertex of $U$.

**Lem.** Given non-$(a, b)$-paintable $G$ and $(a, b)$-gadget $(H, U)$, no $(H, U)$-aug. of $G$ is $(a, b)$-paintable.
Reducing 3-paint-critical family

**Def.** Given a graph $H$ and $U \subseteq V(H)$, we say $(H, U)$ is an $(a, b)$-gadget if $H$ is $(a, b)$-colorable and **every** proper coloring gives the same colors to all vertices of $U$.

**Ex.** $H = P_{2n+1}$, $U = \{v_1, v_{2n+1}\}$, $a = 2m$, $b = m$.

**Def.** Given a graph $G$, a vertex $v \in V(G)$, and an $(a, b)$-gadget $(H, U)$, an $(H, U)$-augmentation of $G$ is obtained from $G + H$ by splitting $v$ into $|U|$ copies, partitioning the edges among those copies, and identifying each copy with a vertex of $U$.

**Lem.** Given non-$(a, b)$-paintable $G$ and $(a, b)$-gadget $(H, U)$, no $(H, U)$-aug. of $G$ is $(a, b)$-paintable.

**Appl.** Any 3-paint-critical graph can be obtained by replacing edges with odd-length paths in one of $C_3, K_{2,4}, \Theta_{1,3,3}, \Theta_{2,2,4}, \diamondsuit\diamondsuit\diamondsuit$, or $\diamondsuit\diamondsuit\diamondsuit\diamondsuit$. 
Reducing 3-paint-critical family

**Def.** Given a graph $H$ and $U \subseteq V(H)$, we say $(H, U)$ is an $(a, b)$-gadget if $H$ is $(a, b)$-colorable and every proper coloring gives the same colors to all vertices of $U$.

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**Lem.** Given non-$(a, b)$-paintable $G$ and $(a, b)$-gadget $(H, U)$, no $(H, U)$-aug. of $G$ is $(a, b)$-paintable.

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• Non-$(2m, m)$-paintability is preserved.
3-Paint-critical $\Rightarrow$ non-$(2m, m)$-paintable

- Already covered $C_3$. 
3-Paint-critical $\implies$ non-$(2m, m)$-paintable

- Already covered $C_3$.

- All other 3-paint-critical graphs are bipartite.
3-Paint-critical $\Rightarrow$ non-$\langle 2m, m \rangle$-paintable

- Already covered $C_3$.

- All other 3-paint-critical graphs are bipartite.

**Lem.** Let $uv \in E(G)$ and $g(u) + g(v) = \max\{f(u), f(v)\}$. If Lister marks $u$ and $v$, then Painter must color $u$ or $v$ to avoid losing.
3-Paint-critical \Rightarrow \text{non-}(2m, m)\text{-paintable}

- Already covered $C_3$.
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**Lem.** Let $uv \in E(G)$ and $g(u) + g(v) = \max\{f(u), f(v)\}$. If Lister marks $u$ and $v$, then Painter must color $u$ or $v$ to avoid losing.

**Cor.** For the $(2m, m)$-paintability game, if $G$ is bipartite and Lister marks $V(G)$, then Painter must color a partite set to avoid losing.
3-Paint-critical $\Rightarrow$ non-$(2m, m)$-paintable

- Already covered $C_3$.
- All other 3-paint-critical graphs are bipartite.

**Lem.** Let $uv \in E(G)$ and $g(u) + g(v) = \max\{f(u), f(v)\}$. If Lister marks $u$ and $v$, then Painter must color $u$ or $v$ to avoid losing.

**Cor.** For the $(2m, m)$-paintability game, if $G$ is bipartite and Lister marks $V(G)$, then Painter must color a partite set to avoid losing.

**Appl.** Let $f(v) = 2m, g(v) = m$, and $G$ be bipartite. If Lister marks $V(G)$ each round until a partite set is missing one color,
3-Paint-critical $\Rightarrow$ non-(2m, m)-paintable

- Already covered $C_3$.

- All other 3-paint-critical graphs are bipartite.

**Lem.** Let $uv \in E(G)$ and $g(u) + g(v) = \max \{f(u), f(v)\}$. If Lister marks $u$ and $v$, then Painter must color $u$ or $v$ to avoid losing.

**Cor.** For the (2m, m)-paintability game, if $G$ is bipartite and Lister marks $V(G)$, then Painter must color a partite set to avoid losing.

**Appl.** Let $f(v) = 2m, g(v) = m$, and $G$ be bipartite. If Lister marks $V(G)$ each round until a partite set is missing one color, then each $v$ has $r + 1$ tokens left, where $r = (#\text{missing colors in the other partite set})$. 
Special case: $C_4$

**Lem.** Lister can win in the following configurations:

- $(r, r)$
- $(r + 1, r)$
- $(r + 1, 1)$
- $(r, 1)$
- $(r + 1, 1)$
Special case: $C_4$

**Lem.** Lister can win in the following configurations:

- $(r, r)$
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- $(r, 1)$
- $(r + 1, 1)$
- $(r + 1, r)$

**Pf. Basis:** $r = 1$

- $(1, 1)$
- $(2, 1)$
- $(2, 1)$
Special case: $C_4$

**Lem.** Lister can win in the following configurations:

- $(r, r)$
- $(r + 1, 1)$
- $(r + 1, 1)$
- $(r + 1, r)$
- $(r + 1, r)$
- $(r + 1, 1)$
- $(r, 1)$
- $(r, 1)$

**Pf.** Case 1:
Special case: $C_4$

**Lem.** Lister can win in the following configurations:

- $(r, r)$
- $(r + 1, r)$
- $(r + 1, 1)$
- $(r, 1)$
- $(r + 1, 1)$

**Pf.** Case 1:

- $(r, r)$
- $(1, 1)$
- $(r + 1, r)$
- $(r + 1, r)$
Special case: \( C_4 \)

**Lem.** Lister can win in the following configurations:

- \((r, r)\)
- \((r + 1, r)\)
- \((r + 1, 1)\)
- \((r, 1)\)
- \((r + 1, r)\)

**Pf.** Case 2:
Special case: $C_4$

**Lem.** Lister can win in the following configurations:

- $(r, r)$
- $(r + 1, r)$
- $(r + 1, 1)$
- $(r + 1, 1)$
- $(r + 1, 1)$
- $(r, r)$
- $(r + 1, r)$
- $(r, r - 1)$
- $(r, 1)$
- $(r + 1, r)$

**Pf.** Case 2:
Special case: $C_4$

**Lem.** Lister can win in the following configurations:

- $(r, r)$
- $(r + 1, 1)$
- $(r, 1)$
- $(r + 1, r)$
- $(r + 1, r)$
- $(r + 1, r)$
- $(r + 1, r)$
- $(r + 1, 1)$

**Pf.** Case 2:

- $(r, r - 1)$
- $(r - 1, 1)$
- $(r, 1)$
- $(r, r - 1)$
Special case: $C_4$

**Lem.** Lister can win in the following configurations:

- $(r, r)$
- $(r + 1, r)$
- $(r + 1, 1)$
- $(r, 1)$
- $(r + 1, 1)$

**Cor.** The following graph is not $(2m, m)$-paintable.
Special case: $C_4$

**Lem.** Lister can win in the following configurations:

- $(r, r)$
- $(r + 1, r)$
- $(r + 1, 1)$
- $(r, 1)$
- $(r + 1, 1)$

**Cor.** The following graph is not $(2m, m)$-paintable.

**Pf.** Lister marks all vertices until one partite set needs one more color.
Special case: $C_4$

**Lem.** Lister can win in the following configurations:

- $(r, r)$
- $(r + 1, r)$
- $(r + 1, 1)$
- $(r, 1)$
- $(r + 1, 1)$

**Cor.** The following graph is not $(2m, m)$-paintable.

**Pf.** Lister marks all vertices until one partite set needs one more color. Then Lister marks the two $3$-vertices, and wins on uncolored $C_4$ by the Lemma.
$K_{2,4}$ and Other Bipartite Cases

After Lister marks all vertices until one partite set is missing one color:

- $(r + 1, 1)$
- $(r + 1, r)$
- $(r + 1, 1)$
After **Lister** marks all vertices until one partite set is missing one color:

\[(r + 1, 1), (r + 1, r), (r, r)\]

If **Painter** colors the top vertex, then **Lister** wins on
$K_{2,4}$ and Other Bipartite Cases

After Lister marks all vertices until one partite set is missing one color:

Otherwise, Lister marks the complement of the first set.
After Lister marks all vertices until one partite set is missing one color:

Using induction on $r$, we're done. (Basis: $\chi_p(K_{2,4}) > 2$)
$K_{2,4}$ and Other Bipartite Cases

After Lister marks all vertices until one partite set is missing one color:

$(r + 1, r)$

$(r + 1, 1)$

$(r + 1, r)$

The other case for $K_{2,4}$ follows the same arguments.
After Lister marks all vertices until one partite set is missing one color:

\[(r + 1, r)\]

\[(r + 1, 1)\]

\[(r + 1, r)\]

The other case for $K_{2,4}$ follows the same arguments.

For the remaining 3-paint-critical graphs:
**$K_{2,4}$ and Other Bipartite Cases**

After *Lister* marks all vertices until one partite set is missing one color:

- $(r + 1, r)$
- $(r + 1, 1)$
- $(r + 1, r)$

The other case for $K_{2,4}$ follows the same arguments.  

For the remaining 3-paint-critical graphs:

- *Lister* marks $V(G)$ until some part is missing one color.
**$K_{2,4}$ and Other Bipartite Cases**

After Lister marks all vertices until one partite set is missing one color:

$$\begin{align*}
(r + 1, r) \\
(r + 1, 1) \\
(r + 1, r)
\end{align*}$$

The other case for $K_{2,4}$ follows the same arguments.

For the remaining 3-paint-critical graphs:
- Lister marks $V(G)$ until some part is missing one color.
- Lister plays to force a $C_4$ from the Lemma.
\(K_{2,4}\) and Other Bipartite Cases

After Lister marks all vertices until one partite set is missing one color:

\[(r + 1, r)\]

\[(r + 1, 1)\]

\[(r + 1, r)\]

The other case for \(K_{2,4}\) follows the same arguments.

For the remaining 3-paint-critical graphs:

- Lister marks \(V(G)\) until some part is missing one color.
- Lister plays to force a \(C_4\) from the Lemma.

\[\Theta_{1,3,3}\]
\[\Theta_{2,2,4}\]
\[C_4 \cdot C_4\]

\[\therefore (2m, m)\)-paintable \(\Rightarrow\) \((2, 1)\)-paintable.
(2m, m)-Paintability of $K_{2,3}$

Given $f, g$, we define vertex names and weights.
(2m, m)-Paintability of $K_{2,3}$

Given $f, g$, we define vertex names and weights.

$B(a_i b_j) = f(a_i) - g(a_i b_j)$
$B(a_i b_j) = f(b_j) - g(a_i b_j)$
(2m, m)-Paintability of $K_{2,3}$

Given $f, g$, we define vertex names and weights.

For an edge $e$ and nonempty set $D$ of edges disjoint from $e$, we require

\[ w_A(e) + w_A(D) \geq 0 \]
\[ w_B(e) + w_B(D) \geq 0. \]

\[ w_A(a_ib_j) = f(a_i) - g(a_ib_j) \]
\[ w_B(a_ib_j) = f(b_j) - g(a_ib_j) \]
Given $f, g$, we define vertex names and weights.

For an edge $e$ and nonempty set $D$ of edges disjoint from $e$, we require

\[ w_A(e) + w_A(D) \geq 0 \]
\[ w_B(e) + w_B(D) \geq 0. \]

A vertex $v$ is forced if $f(v) = g(v)$.

\[ w_A(a_ib_j) = f(a_i) - g(a_ib_j) \]
\[ w_B(a_ib_j) = f(b_j) - g(a_ib_j) \]
(2m, m)-Paintability of \(K_{2, 3}\)

Given \(f, g\), we define vertex names and weights.

For an edge \(e\) and nonempty set \(D\) of edges disjoint from \(e\), we require

\[
\begin{align*}
\omega_A(e) + \omega_A(D) &\geq 0 \\
\omega_B(e) + \omega_B(D) &\geq 0.
\end{align*}
\]

A vertex \(v\) is forced if \(f(v) = g(v)\).

If Lister marks \(M\) and no vertex is forced, then Painter plays according to the following strategy:

\[
\begin{align*}
\omega_A(a_i b_j) &= f(a_i) - g(a_i b_j) \\
\omega_B(a_i b_j) &= f(b_j) - g(a_i b_j)
\end{align*}
\]
(2m, m)-Paintability of $K_{2,3}$

Given $f, g$, we define vertex names and weights.

For an edge $e$ and nonempty set $D$ of edges disjoint from $e$, we require

\begin{align*}
  w_A(e) + w_A(D) &\geq 0 \\
  w_B(e) + w_B(D) &\geq 0.
\end{align*}

A vertex $\nu$ is forced if $f(\nu) = g(\nu)$.

If Lister marks $M$ and no vertex is forced, then Painter plays according to the following strategy:

1) $a_i \in M, b_j \notin M, w_B(a_ib_j) < 0$ \quad \Rightarrow \quad Color M \cap A,

\begin{align*}
  w_A(a_ib_j) &= f(a_i) - g(a_ib_j) \\
  w_B(a_ib_j) &= f(b_j) - g(a_ib_j)
\end{align*}
((2m, m)-Paintability of $K_{2, 3}$)

Given $f, g$, we define vertex names and weights.  

For an edge $e$ and nonempty set $D$ of edges disjoint from $e$, we require 

(*) $w_A(e) + w_A(D) \geq 0$

$w_B(e) + w_B(D) \geq 0$.

A vertex $v$ is forced if $f(v) = g(v)$.

If Lister marks $M$ and no vertex is forced, then Painter plays according to the following strategy:

1) $a_i \in M, b_j \notin M, w_B(a_ib_j) < 0$  \hspace{1cm} \Rightarrow \hspace{1cm} \text{Color } M \cap A,

2) $a_i \notin M, b_j \in M, w_A(a_ib_j) < 0$  \hspace{1cm} \Rightarrow \hspace{1cm} \text{Color } M \cap B,

\[
\begin{align*}
w_A(a_ib_j) &= f(a_i) - g(a_ib_j) \\
w_B(a_ib_j) &= f(b_j) - g(a_ib_j)
\end{align*}
\]
(2m, m)-Paintability of \( K_{2,3} \)

Given \( f, g \), we define vertex names and weights.

For an edge \( e \) and nonempty set \( D \) of edges disjoint from \( e \), we require

\[
\begin{align*}
\omega_A(e) + \omega_A(D) &\geq 0 \\
\omega_B(e) + \omega_B(D) &\geq 0.
\end{align*}
\]

A vertex \( v \) is forced if \( f(v) = g(v) \).

If Lister marks \( M \) and no vertex is forced, then Painter plays according to the following strategy:

1) \( a_i \in M, b_j \notin M, \omega_B(a_i b_j) < 0 \) \( \Rightarrow \) Color \( M \cap A \),
2) \( a_i \notin M, b_j \in M, \omega_A(a_i b_j) < 0 \) \( \Rightarrow \) Color \( M \cap B \),
3) \( |M \cap A| \geq |M \cap B| \) \( \Rightarrow \) Color \( M \cap A \),

where

\[
\begin{align*}
w_A(a_i b_j) &= f(a_i) - g(a_i b_j) \\
w_B(a_i b_j) &= f(b_j) - g(a_i b_j)
\end{align*}
\]
(2m, m)-Paintability of $K_{2,3}$

Given $f, g$, we define vertex names and weights.

For an edge $e$ and nonempty set $D$ of edges disjoint from $e$, we require

\begin{align*}
\text{(1)} \quad w_A(e) + w_A(D) &\geq 0 \\
\text{(2)} \quad w_B(e) + w_B(D) &\geq 0.
\end{align*}

A vertex $\nu$ is forced if $f(\nu) = g(\nu)$.

If Lister marks $M$ and no vertex is forced, then Painter plays according to the following strategy:

1) $a_i \in M, b_j \notin M, w_B(a_ib_j) < 0 \Rightarrow \text{Color } M \cap A$,

2) $a_i \notin M, b_j \in M, w_A(a_ib_j) < 0 \Rightarrow \text{Color } M \cap B$,

3) $|M \cap A| \geq |M \cap B| \Rightarrow \text{Color } M \cap A$,  

$w_A(a_ib_j) = f(a_i) - g(a_ib_j)$

$w_B(a_ib_j) = f(b_j) - g(a_ib_j)$
Given $f, g$, we define vertex names and weights.

$$w_A(a_ib_j) = f(a_i) - g(a_ib_j)$$
$$w_B(a_ib_j) = f(b_j) - g(a_ib_j)$$

For an edge $e$ and nonempty set $D$ of edges disjoint from $e$, we require

$$w_A(e) + w_A(D) \geq 0$$
$$w_B(e) + w_B(D) \geq 0.$$

A vertex $v$ is forced if $f(v) = g(v)$.

If Lister marks $M$ and no vertex is forced, then Painter plays according to the following strategy:

1) $a_i \in M$, $b_j \notin M$, $w_B(a_ib_j) < 0$ $\Rightarrow$ Color $M \cap A$,

2) $a_i \notin M$, $b_j \in M$, $w_A(a_ib_j) < 0$ $\Rightarrow$ Color $M \cap B$,

3) $|M \cap A| \geq |M \cap B|$ $\Rightarrow$ Color $M \cap A$,

4) Otherwise, $\Rightarrow$ Color $M \cap B$.
(2m, m)-Paintability of $K_{2,3}$

Given $f, g$, we define vertex names and weights.

For an edge $e$ and nonempty set $D$ of edges disjoint from $e$, we require

\[ w_A(e) + w_A(D) \geq 0 \]
\[ w_B(e) + w_B(D) \geq 0. \]

A vertex $v$ is forced if $f(v) = g(v)$.

If Lister marks $M$ and no vertex is forced, then Painter plays according to the following strategy:

1) $a_i \in M, b_j \notin M, w_B(a_ib_j) < 0 \Rightarrow$ Color $M \cap A$,
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3) $|M \cap A| \geq |M \cap B| \Rightarrow$ Color $M \cap A$,
4) Otherwise, Color $M \cap B$.

Always, $(*)$ preserved and $\max\{w_A(e), w_B(e)\} \geq 0$. 

\[ w_A(a_ib_j) = f(a_i) - g(a_ib_j) \]
\[ w_B(a_ib_j) = f(b_j) - g(a_ib_j) \]
(2m, m)-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.
(2m, m)-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.

**Case 1:** $a_1$ is forced.

$w_A = -y_i$
$w_B = x_i - a$

$(x_i + y_i, y_i)$

$(c + d, d)$

$(a, a)$
(2m, m)-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.

**Case 1:** $a_1$ is forced.

$$\max\{w_A(e), w_B(e)\} \geq 0 \Rightarrow x_i \geq a$$

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$(x_i + y_i, y_i)$
(2m, m)-Paintability of $K_{2, 3}$ (Cont.)

Painter uses strategy until a vertex is forced.

**Case 1:** $a_1$ is forced.

$\max\{w_A(e), w_B(e)\} \geq 0 \Rightarrow x_i \geq a$

$w_A(e) = c - y_1$, $w_A(D) = -y_2 - y_3$

$\Rightarrow c \geq y_1 + y_2 + y_3$
(2m, m)-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.

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Lister marks $N[a_1]$ for $a$ rounds.
(2m, m)-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.

Case 1: $a_1$ is forced.

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\]

\[
w_A(e) = c - y_1, \quad w_A(D) = -y_2 - y_3
\]

\[
\Rightarrow c \geq y_1 + y_2 + y_3
\]

Lister marks $N[a_1]$ for $a$ rounds.

Painter wins by degeneracy.

(#tokens on $a_2$ is at least $d + \sum y_i$.)
(2m, m)-Paintability of $K_{2,3}$ (Cont.)

**Painter** uses strategy until a vertex is **forced**.

**Case 2**: $b_1$ is forced.

\[
\begin{align*}
w_B &= -y_i \\
(b, b) &\quad \text{(Painted)} \\
(c_i + d_i, d_i) &\quad \text{(Painted)} \\
(w_A = x_i - b) &\quad \text{(Not painted)} \\
(x_i + y_i, y_i) &\quad \text{(Painted)}
\end{align*}
\]
(2m, m)-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.

**Case 2:** $b_1$ is forced.

$$\max\{w_A(e), w_B(e)\} \geq 0 \Rightarrow x_i \geq b$$
(2m, m)-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.

**Case 2:** $b_1$ is forced.

$max\{w_A(e), w_B(e)\} \geq 0 \Rightarrow x_i \geq b
w_B(e) = c_2 - y_1, w_B(D) = -y_2
\Rightarrow c_2 \geq y_1 + y_2$
(2m, m)-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.

**Case 2:** $b_1$ is forced.

$\max \{w_A(e), w_B(e)\} \geq 0 \Rightarrow x_i \geq b$

- $w_B(e) = c_2 - y_1$, $w_B(D) = -y_2$
- $\Rightarrow c_2 \geq y_1 + y_2$

Similarly, $c_3 \geq y_1 + y_2$
(2m, m)-Paintability of $K_{2,3}$ (Cont.)

**Painter** uses strategy until a vertex is forced.

**Case 2:** $b_1$ is forced.
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\Rightarrow c_2 \geq y_1 + y_2
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Similarly, $c_3 \geq y_1 + y_2$

**Lister** marks $N[b_1]$ for $b$ rounds.
(2m, m)-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.

**Case 2:** $b_1$ is forced.

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Lister marks $N[b_1]$ for $b$ rounds.

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(#tokens on $b_j$ is at least $d_j + \sum y_i$.)
(2m, m)-Paintability of $K_{2,3}$ (Cont.)

Painter uses strategy until a vertex is forced.

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(2m, m)-Paintability of $K_{2,3}$ (Cont.)

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Similarly, $c_3 \geq y_1 + y_2$

Lister marks $N[b_1]$ for $b$ rounds.

Painter wins by degeneracy.

(#tokens on $b_j$ is at least $d_j + \sum y_i$.)

$\therefore (2, 1)$-paintable $\Rightarrow (2m, m)$-paintable.
Open Questions

**Ques.** Given 3-paint-critical $G$ and $m > 1$, what is $\min t$ such that $G$ is $(t, m)$-paintable?
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**Conj.** $G \notin \{C_{2k+1}, K_k\} \implies G$ is $(\Delta(G)m, m)$-paintable.
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Thank You!