Online Scheduling and Paintability

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Joint work with
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List Coloring (Graph Choosability)

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**Goal:** Consider an online version of choosability.
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Worst-case analysis is modeled by the following game:
Lister/Painter Game (Schauz [2009])

**Two players:** Lister and Painter on a graph G with a positive number of tokens at each vertex.
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- An adaptive Lister, responding to Painter’s earlier moves, may do better.
Example Game

Let’s play the **Lister/Painter** game on $\Theta_{2,2,4}$.
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![Diagram of a game setup with nodes labeled 1, 2, 2, 2, 2, and connections between them.]*
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![Game Board Diagram]
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\[
\begin{array}{ccc}
\text{0} & & \text{0} \\
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Conclude: Lister wins on $\Theta_{2,2,4}$ when each vertex has 2 tokens.
Definitions

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**Obs.** \( k \)-paintable \( \Rightarrow k \)-choosable \( \Rightarrow k \)-colorable. Thus \( \chi(G) \leq \chi_l(G) \leq \chi_p(G) \) for all \( G \).
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When $G$ is connected and not in $\{K_n, C_{2t+1}\}$,

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When $G$ is bipartite,
- $G$ is $\Delta(G)$-edge-colorable \text{(König [1916])}
- $G$ is $\Delta(G)$-edge-choosable \text{(Galvin [1995])}
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The line graph of $K_k$ is
- $k$-colorable (Exercise)
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- 5 teams (10 games total)
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Def. The join of $G$ and $H$, denoted $G \diamond H$, is the disjoint union $G + H$ plus edges joining all of $V(G)$ to all of $V(H)$.
**Tools**

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**Def.** The join of \( G \) and \( H \), denoted \( G \oplus H \), is the disjoint union \( G + H \) plus edges joining all of \( V(G) \) to all of \( V(H) \).

**Thm.** (CLMPTW) If \( G \) is \( k \)-paintable and \( |V(G)| \leq \frac{t}{t-1} k \), then \( G \oplus \overline{K}_t \) is \((k + 1)\)-paintable.
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**Pf. Idea:** Painter uses a $k$-paintability strategy $S$ on $G$, ignoring the added $t$-set $T$, until a special round where $M \cap T$ is colored instead. Each $v \in T$ has a token left, and $G$ can be finished with the extra tokens in $V(G)$. 
Ohba’s Conjecture

**Def.** $G$ is chromatic-choosable if $\chi_l(G) = \chi(G)$.

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Complete Bipartite Graphs

**Thm.** (Vizing [1976]) $K_{k,r}$ is $k$-choosable $\iff r < k^k$. 
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**Thm.** (CLMPTW) Consider $K_{k,r}$ with parts $X$ of size $k$ and $Y$ of size $r$. If each vertex of $Y$ has $k$ tokens, then Painter has a winning strategy $\iff r < \prod_{i=1}^{k} t_i$, where $t_1, \ldots, t_k$ are the token counts in $X$. 

![Diagram of a complete bipartite graph $K_{k,r}$ with parts $X$ and $Y$, where $X$ has size $k$ and $Y$ has size $r$. Each vertex in $Y$ has $k$ tokens, and Painter has a winning strategy if $r < \prod_{i=1}^{k} t_i$, where $t_1, \ldots, t_k$ are the token counts in $X$.](image)
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$k$-paintability for $K_{k,r}$

**Thm.** (CLMPTW) Consider $K_{k,r}$ with $|X| = k$ and $|Y| = r$. If $f(y) = k$ for $y \in Y$ and $f(x_i) = t_i$ for $x_i \in X$, then Painter has a winning strategy $\iff r < \prod_{i=1}^{k} t_i$. 
**k-paintability for $K_{k,r}$**

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Painter has a winning strategy $\iff r < \prod_{i=1}^{k} t_i$.

**Pf.** $r = \prod t_i \Rightarrow K_{k,r}$ is not $f$-choosable.

Let $L(x_i) = U_i$ with $|U_i| = t_i$ and pairwise disjoint.

Let $\{L(y) : y \in Y\} = U_1 \times \cdots \times U_k$. 
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$r < \prod t_i \Rightarrow \text{Painter wins.}$
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**Case 1:** $q < \prod_{i=1}^{k-1} t_i$. Painter colors $x_k$. $Y - M$ is degenerate; apply ind. hyp. to $(X - x_k) \cup (M \cap Y)$. 

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**Case 2:** $q \geq \prod_{i=1}^{k-1} t_i$. Painter colors $M \cap Y$.
$|Y - M| < \prod t_i - q \leq \prod_{i=1}^{k-1} t_i(t_k - 1)$; ind. hyp. applies!
Open Question

**Ques.** Can $\chi_p(G) - \chi_l(G) > 1$?
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Graphs to consider:

**Possibility 1:** Complete bipartite graphs
- $\chi_l(K_{k,k}) \leq \lg k - \left(\frac{1}{2} + o(1)\right) \lg \lg k$ (Alon)
- $\chi_p(K_{k,k}) \leq \lg k$ (KKLZ [2012])
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Possibility 2: Complete multipartite graphs

$\chi_\ell(K_{3*k}) = \left\lceil \frac{4k-1}{3} \right\rceil$ (Kierstead [2000])

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Ques. What is $\min \{r: K_{k+j,r} \text{ is not } k\text{-paintable}\}$?
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Hard to compute for \( j > 0 \)!
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**Thank You!**