

# Bootstrap Percolation on Hexagonal Lattices

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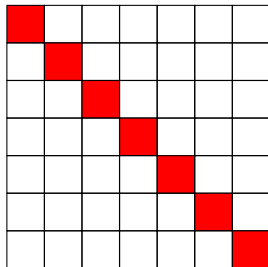
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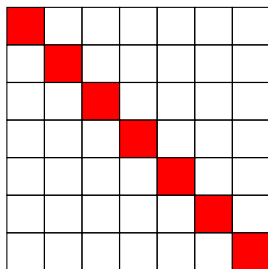
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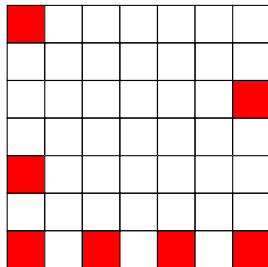
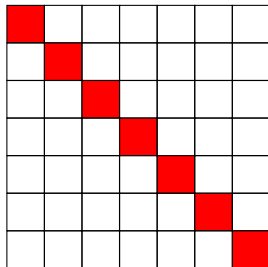
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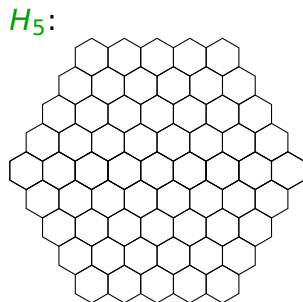
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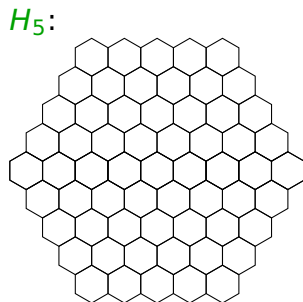
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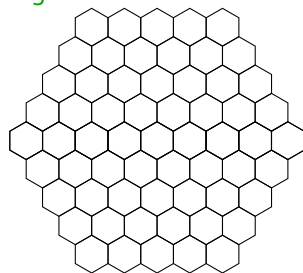
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| $n$ | min         | max       |
|-----|-------------|-----------|
| 1   | 0           | 0         |
| 2   | 2           | 2         |
| 3   | 6           | 6         |
| 4   | 9           | 19        |
| 5   | 10          | 33        |
| 6   | $\leq 18^*$ | $\geq 48$ |
| 7   | $\leq 22^*$ | $\geq 59$ |
| 8   | $\leq 21$   | ?         |

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