Name: ______________________________

Worksheet #22

Math 231 AD1

Starred questions are from the professor's worksheet and are of particular importance.

Suggested exercies: Section 11.11 (page 774–775), 13–26.

Warm-up

1. Give the Maclaurin series for $-\sin(x)$. Using integration, find a power series for $\cos(x)$.
2. Using Taylor's Inequality, estimate $\cos(0.1)$ to within 10^{-6}. Is your answer reasonable?

Taylor Series Applications

3. *(Diff. Eq.) Power series can be used to solve differential equations. Suppose that a country has at time t a population $p(t)$, and that the rate of change of the population is given by $p'(t) = 2p(t)$.
   (a) If we represent that population by a power series $p_0 + p_1 t + p_2 t^2 + \cdots$, find from the differential equation a relation between the coefficients p_n.
   (b) Find the power series if $p_0 = 1$.
   (c) Identify the power series as a Taylor expansion of some function $f(t)$.
4. *Approximate $f(x) = \sqrt{x}$ by a Taylor polynomial of degree 2 centered at $x = 4$. Then use Taylor's Inequality to estimate the accuracy of the approximation for $4 \leq x \leq 4.2$.
5. *An electric charge q at $x = 0$ produces an electric field $E(x) = \frac{q}{x^2}$. If we have two opposite electric charges situated at $x = -d$ (charge $-q$) and $x = 0$ (charge $+q$), then the total electric field produced at D is $E(D) = \dfrac{q}{D^2} - \dfrac{q}{(D+d)^2}$.
   (a) Find the expansion of $\dfrac{1}{(D+d)^2} = \dfrac{1}{D^2(1+\frac{d}{D})^2}$ in positive powers of $\frac{d}{D}$.
   (b) Use this to find an approximation of $E(D) \approx \frac{2qd}{D^3}$ for large values of D. This approximation is used often in Physics.

The Sum of Inverse Squares

6. We are now going to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.
   (a) Find the Taylor series for $f(x) = \frac{1}{\sqrt{1-x^2}}$ centered at 0.
   (b) Use this series to find the series for $\sin^{-1}(x)$.
   (c) Show that $\int_0^1 \frac{\arcsin(x)}{\sqrt{1-x^2}} dx = \frac{\pi^2}{8}$. (Hint: don't use power series.)
   (d) Using the fact that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n+1}(x)dx = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n+1}$, evaluate $\displaystyle\int_0^1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{\sqrt{1-x^2}}dx$.
   (e) Now show that $\displaystyle\int_0^1 \frac{\arcsin(x)}{\sqrt{1-x^2}}dx = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.
   (f) Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.