Name: ____________________________________

Worksheet #18

Math 231 AD1

Starred questions are from the professor's worksheet and are of particular importance.
Suggested exercies: Section 11.8 (page 745), 3–28.

Introduction

1. *Given a power series $\sum_{n=0}^{\infty} a_n x^n$, suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.
   What is the radius of convergence of this power series?
2. *If for $\sum_{n=0}^{\infty} a_n(x-c)^n$ we have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then find the radius of convergence?
3. Here's a power series *centered at c*: $\sum_{n=0}^{\infty} a_n(x-c)^n$. It can do one of three things:
   1) For some R, the series diverges for $|x-a| > R$ and converges for $|x-a| < R$.
      (R is the *radius of convergence*)
   2) The series converges only when $x = a$. We say $R = 0$ in this case.
   3) The series converges for all x. We say $R = \infty$ in this case.

   (a) How would you determine which of the 3 possibilities is true for a given series?

   (b) Why is there no conclusion for when $|x-a| = R$ in the first possibility?

   (c) For each of the 3 possibilities, write down the interval of convergence.

Power Series

4. *Suppose $a_n = (-1)^n \frac{n^2}{2^n}$. What is $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$?
   What is the *interval* of convergence of $\sum_{n=0}^{\infty} a_n(x-2)^n$?
5. *Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $x = -4$, but diverges for $x = 6$.

   (a) What can you say about the convergence of $f(x)$ at $x = 2$? $x = -7$?

   (b) What can you say about the convergence of $\sum_{n=0}^{\infty} a_n$?

   (c) What can you say about the convergence of $\sum_{n=0}^{\infty}(-1)^n a_n 3^n$?
6. *Find the Radius and Interval of Convergence for each of the following power series:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (c) $\displaystyle\sum_{n=0}^{\infty} \sqrt{n}x^n$ (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(x-3)^n}{2n+1}$

Preview (more power series)

7. Consider the series $\sum_{n=0}^{\infty}(x-2)^n$
   (a) *For what values of x does the series absolutely converge?
   (b) What is the value of the series (in terms of x) when it converges?
   (c) Let $f(x) = \sum_{n=0}^{\infty}(x-2)^n$. When the series converges, what is the value of $f'(x)$?

8. Consider the following series: Let $f(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$.
   (a) *For what values of x does the series absolutely converge?
   (b) Compute the first four derivatives of $f(x)$. Try to simplify $f''(x)$ and $f^{(4)}(x)$. (Hint: Write out several terms of each series and differentiate term-by-term.)
   (c) How are $f(x)$, $f''(x)$, and $f^{(4)}(x)$ related? What about $f'(x)$ and $f^{(3)}(x)$?
   (d) What *familiar* function do you think $f(x)$ is? What about $f'(x)$?

9. Recall that $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.
   (a) Using the previous problem, show that $e^{ix} = \cos(x) + i\sin(x)$ where $i^2 = -1$.
   (b) Conclude Euler's Identity: $e^{i\pi} = -1$.