Worksheet #19 Solutions

Math 221

1. In a murder investigation, the temperature of the corpse was 32.5°C at 1:30 PM and 30.3°C an hour later. Normal body temperature is 37.0°C and the temperature of the surroundings was 20.0°C. When did the murder take place?

Newton's law of cooling states $\frac{dT}{dt} = k(T - T_s)$ where T is the temperature at time t, k is some constant and T_s is the (constant) temperature of the surrounding air. To make this *differential equation* easier to study, we introduce a new function $y(t)$ that represents the *difference* between the temperature T and the surrounding temperature T_s:

$$y(t) = T(t) - T_s.$$

Newton's law of cooling now says $\frac{dy}{dt} = ky$. Theorem 2 (page 237) tells us that the only solution to this differential equation is of the form $y(t) = y(0)e^{kt}$. (The whole reason for using $y(t)$ instead of $T(t)$ is to be able to use Theorem 2.)

We are given that the initial temperature $T(0) = 37$°C and ambient temperature $T_s = 20$°C. Thus $y(t) = (37 - 20)e^{kt} = 17e^{kt}$.

What we want to know is which value of t corresponds to 1:30 PM. Let this value be x, and so $x+1$ corresponds to 2:30 PM. Then $y(x) = (32.5 - 20) = 17e^{kx}$. Also $y(x+1) = (30.3 - 20) = 17e^{k(x+1)}$. We have two equations and two unknowns (k and x), so we solve this system by dividing the first equation by the second to get

$$\frac{12.5}{10.3} = \frac{17e^{kx}}{17e^{k(x+1)}} = \frac{1}{e^k}.$$

Solving for k, we get $k = \ln(\frac{10.3}{12.5})$. Now we plug this into one of our original equations to get

$$12.5 = 17e^{\ln(\frac{10.3}{12.5})x}.$$

Solving for x, we get $x = \frac{\ln(\frac{12.5}{17})}{\ln(\frac{10.3}{12.5})} \approx 1.59$ hours, or about 95 minutes. Thus the murder took place at about 11:55 AM, just in time for lunch.

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2. You are blowing up a spherical balloon by blowing air into it at a rate of 8 cubic inches per second. How fast is the radius of the balloon increasing when it is 1 inch? What about when it is 5 inches?

We know that $V = \frac{4}{3}\pi r^3$, but because the volume (and also the radius) are changing, we think of this as $V(t) = \frac{4}{3}\pi r(t)^3$. Take the time derivative of both sides to get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

We are given that $\frac{dV}{dt} = 8$, and we want to know $\frac{dr}{dt}$ when $r = 1$ and $r = 5$. Plugging everything in, we get $8 = 4\pi(1^2)\dfrac{dr}{dt}$ and $8 - 4\pi(5^2)\dfrac{dr}{dt}$, respectively. Solving for $\frac{dr}{dt}$, we have $\dfrac{dr}{dt} = \dfrac{2}{\pi}$ in/sec and $\dfrac{dr}{dt} = \dfrac{2}{25\pi}$, respectively.

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3. Find the equation of the tangent line to the curve $xy + x + y + \sin(xy) = 1$ at the point $(0, 1)$.

Since we want to find the equation of a line tangent to the given curve, we need two pieces of information: the slope of the line, and a point on the line. We are given the point on the line of $(0, 1)$. Thus our final answer will have the form $y - 1 = m(x - 0)$.

All we need to do is find the slope of the tangent line. Slope is $\frac{\text{rise}}{\text{run}}$, so we need to find $\frac{dy}{dx}$. Taking the derivative of the curve with respect to x, we get

$$(1 \cdot y + x \cdot \frac{dy}{dx}) + 1 + \frac{dy}{dx} + \cos(xy)(y + x\frac{dy}{dx}) = 0.$$

Rearrange the equation so all $\frac{dy}{dx}$ terms are on the same side, factor $\frac{dy}{dx}$ out, and solve:

$$\begin{aligned} x\frac{dy}{dx} + \frac{dy}{dx} + x\frac{dy}{dx}\cos(xy) &= -(y + 1 + y\cos(xy)) \\ \frac{dy}{dx}(x + 1 + x\cos(xy)) &= -(y + 1 + y\cos(xy)) \\ \frac{dy}{dx} &= -\frac{y + 1 + y\cos(xy)}{x + 1 + x\cos(xy)}. \end{aligned}$$

Setting $x = 0$ and $y = 1$, we get $m = -3$. So the final answer (equation of the tangent line) is $y = -3x + 1$.

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4. Show that the sum of a positive number with its reciprocal is always at least 2.

Let $f(x) = x + \frac{1}{x}$ for $x > 0$. To show that $f(x) \geq 2$, we show that the *minimum* of $f(x)$ is at least 2. To find the minimum, we use the first derivative of y:

$$f'(x) = 1 - x^{-2} = \frac{x^2 - 1}{x^2}.$$

Because we are assuming $x > 0$, the only critical point is $x = 1$. Because $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$, we have that $x = 1$ is a local min, and because there are no other critical points, it is also the global minimum. Therefore $f(x) \geq 2$ for all $x > 0$.

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5. Determine the minimal distance from a point on the graph of $y = 4 - x^2$ to the origin.

Let $D(x)$ be the distance from the origin to the point (x, y) on the given curve. Thus $D(x) = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + (4-x^2)^2}$. We want to minimize $D(x)$, so we take the first derivative:

$$D'(x) = \frac{2x + 2(4-x^2)(-2x)}{2\sqrt{x^2 + (4-x^2)^2}}.$$

The denominator of $D'(x)$ is never negative, so our only critical points occur when $2x - 4x(4 - x^2) = 0$. This gives us critical points $x = 0, \pm\sqrt{\frac{7}{2}}$. Now we check when $D'(x)$ is positive or negative in the following chart:

-	+	-	+
$-\sqrt{\frac{7}{2}}$	0	$\sqrt{\frac{7}{2}}$	

This table shows us that $\pm\sqrt{\frac{7}{2}}$ are the local minima and 0 is a local max. The value of $D(x)$ at both of the local minima points is $\frac{\sqrt{15}}{2}$, and we don't care about the local max.

Because $\lim_{x\to\infty} D(x) = \infty$ and $\lim_{x\to-\infty} D(x) = \infty$, we have that $\frac{\sqrt{15}}{2}$ is the global minimum for $D(x)$.

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6. Find the critical points of $f(x) = \frac{x-1}{x^2+1}$ and determine if they are local or global maxima or minima.

To determine the critical points, we need to find all the places where $f'(x)$ is zero or undefined.

$$f'(x) = \frac{(x^2+1)\cdot 1 - 2x(x-1)}{(x^2+1)^2} = -\frac{x^2 - 2x - 1}{(x^2+1)^2}.$$

Since $(x^2+1)^2$ is never zero, we only need to find where $f'(x) = 0$. Using the quadratic equation, we find roots at $x = 1 \pm \sqrt{2}$. Now we check when $f'(x)$ is postive or negative in the following chart:

+	-	+
$1-\sqrt{2}$	$1+\sqrt{2}$	

This table shows us that $1 - \sqrt{2}$ is a local max and $1 + \sqrt{2}$ is a local min. The values of $f(x)$ at these points are $-\frac{1}{2} - \frac{1}{\sqrt{2}}$ and $-\frac{1}{2} + \frac{1}{\sqrt{2}}$, respectively. (Approximately, these values are -1.2 and 0.2.)

To determine the global max and global min, we need to also check $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. In each case, the x^2 term in the denominator grows faster than the numerator, so both limits are 0. Therefore the local min and max are also the global min and max.

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7. Determine the limit if it exists.

1. $\lim\limits_{x\to\infty} \dfrac{xe^x}{e^{2x}+3}$
   First note that if you try to plug in you get $\frac{\infty}{\infty}$, which is an indeterminate form, so we need to do more work. Factor out the dominating term (e^x) on both top and bottom.

$$\lim_{x\to\infty} \frac{xe^x}{e^{2x}+3} = \lim_{x\to\infty} \frac{e^x}{e^x}\frac{x}{e^x+3e^{-x}} = \lim_{x\to\infty} \frac{x}{e^x} = 0.$$

2. $\lim\limits_{x\to 0^+} (\sin x)(\ln x)$
   First note that if you try to plug in you get $0\cdot(-\infty)$, which is an indeterminate form, so we need to do more work. Put the $\sin(x)$ term in the denominator as $\csc(x)$ and use l'Hopital's Rule.

$$\lim_{x\to 0^+} \sin(x)\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\csc(x)} \overset{L}{=} \lim_{x\to 0^+} \frac{x^{-1}}{-\csc(x)\cot(x)} = \lim_{x\to 0^+} \frac{\tan(x)\sin(x)}{x}.$$

   From here, I can either use l'Hopital's Rule again to get rid of the x in the denominator, or I can recall that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. This leaves me with $\lim_{x\to 0^+} \tan(x) = 0$.

3. $\lim\limits_{x\to 0^+} x^{\sqrt{x}}$
   First note that if you try to plug in you get 0^0, which is an indeterminate form, so we need to do more work. We use logarithms to get rid of the exponent, and then use l'Hopital's Rule with $\ln(x)$ in the numerator.

$$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\ln(x^{\sqrt{x}})} = \lim_{x\to 0^+} e^{\sqrt{x}\ln(x)} = \lim_{x\to 0^+} e^{\frac{\ln(x)}{1/\sqrt{x}}} \overset{L}{=} \lim_{x\to 0^+} e^{\frac{1/x}{-0.5x^{-3/2}}} = \lim_{x\to 0^+} e^{-2\sqrt{x}} = 1$$