15 Sept 2014

Math 181

Ramsey Theory: We're going to talk about Ramsey theory as giving a red–blue coloring of the edges of a complete graph. The *Ramsey number* $R(p, q)$ is the smallest number of vertices in a complete graph that cannot avoid having either a red K_p or a blue K_q.

How hard can it be? Last time, I mentioned Erdős' cute story that if aliens threatened to blow up the Earth unless we compute a particular Ramsey number. He said that if we were asked to compute $R(5, 5)$, *maybe* we could do it in a year if we devote ALL computing power in the planet to this one task. The follow-up was that if we were asked for $R(6, 6)$, we would be better off trying to fight the aliens.

So far, we know that $43 \leq R(5, 5) \leq 49$. To show that it is NOT 43 (and thus must be at least 44), all one needs to do is give a good red–blue coloring of K_{43} that avoids a K_5 in either color.

To show why this is difficult, let's do some counting.

1. How many edges are in K_{43}? For a single edge, there are n choices for the first endpoint and $n - 1$ choices for the second. This counts every edge twice, so we must divide by 2. In general K_n has $n(n - 1)/2$ edges, so K_{43} has 903 edges.
2. How many ways are there to 2-color the edges of K_{43}? Each of the 903 edges has 2 choices, so the total number of colorings is $2 \cdot 2 \cdot 2 \cdots 2$ with 903 factors of 2. We can rewrite this number as 2^{903}. In scientific notation, this is 6.8×10^{271}.
3. OK, so what? Well, an upper bound on the number of particles in the observable universe is 10^{82}. Also, the estimated upper bound on the age of the universe in seconds is 4.35×10^{17}. Let's round it up to 10^{18} to get rid of the 4.35. So if **every particle** tried a coloring **each second** since the big bang, we would have computed 10^{100} (a googol) different colorings.
4. Thinking about how exponents work 10^{100} is NOT between a half and a third of 10^{271}. In fact, even if inside every particle was an entire UNIVERSE the size of ours doing the exact same "one coloring per particle per second", we would STILL only compute 10^{181} colorings.
5. To reach the massive 10^{271} number, we would require a universe within a universe within every particle of our universe and to **double** the age of our universe.

Of course, you could get lucky and find a good red–blue coloring early in this process. But to show that $R(5, 5) = 43$, you need to show that **every** red–blue coloring of K_{44} has either a red K_5 or a blue K_5. There are 5.9×10^{284} such colorings.

Do you now see why Erdős thought it was hopeless if we were asked for $R(6,6)$?

Disclaimer: To be perfectly accurate, I should say a few more things.

1. Not ALL of these colorings are distinct. Some of them "look the same" if you relabel vertices. Our example with K_5 avoiding red and blue triangles can be drawn in several different ways, but they can all be described as a red 5-cycle and a blue 5-cycle. To check all the red–blue colorings of K_{43}, mathematicians would do best to not check the same "type" of coloring multiple times.
2. This would greatly reduce the number of colorings to check, but the process itself of trying to be sure not to check the same thing twice adds complexity and time to the computation.
3. Similarly, it is not straightforward to determine whether or not a particular coloring contains a red K_5 or a blue K_5, so that part of the computation is also difficult.

An Existence Proof: Because of how complex these Ramsey-type structures are, it is not immediately obvious that $R(p,q)$ even *exists*! Maybe there are some numbers for p and q for which I can **always** give a red–blue coloring that avoid having a red K_p or a blue K_q. It turns out that this is not the case, and the proof is nice.

Theorem 1: $R(p,q)$ is *finite*. In particular $R(p,q) \leq R(p-1,q) + R(p,q-1)$.

Proof. Suppose that there exist values for p and q where $R(p,q)$ is *infinite*. There may be many such pairs, so choose p and q to be a pair with smallest sum $p+q$. This means that if two numbers a and b have $a+b < p+q$, then $R(a,b)$ is some finite number.

Let $N = R(p-1,q)+R(p,q-1)$. Both of those pairs have sum $p+q-1$, which is smaller than $p+q$, so those Ramsey numbers exist. Let's consider a red–blue coloring of K_N. Choose some arbitrary vertex v, and look at the colors of the edges coming out of v. There are $R(p-1,q) + R(p,q-1) - 1$ edges coming out of it.

We now use the Pigeonhole Principle with two classes (red and blue). There are $R(p-1,q)$ red edges or there are $R(p,q-1)$ blue edges coming out of v.

Suppose there are $R(p-1,q)$ red edges coming out of v. First note that $R(p-1,q)$ means that among the vertices that are connected by red edges to v, there must be either a red K_{p-1} or a blue K_q. If there is a blue K_q, then we are done. And if there is a red K_{p-1}, then those edges together with the red edges to v, we have found a red K_p.

Similarly, if we have $R(p,q-1)$ blue edges coming out of v, then within those vertices, there is either a red K_p (and we're done) or a blue K_{q-1}, and with the blue edges to v, we're done.

Therefore, we **must** have either a red K_p or a blue K_q, so $R(p,q) \leq R(p-1,q) + R(p,q-1)$.