Math 241, Spring 2007, Merit Worksheet 17

1. Suppose that $D$ is the disk with $x^2 + y^2 \leq 1$. Find (without integrating)

(a) $\int \int_D \sqrt{1 - x^2 - y^2} \, dA$

(b) $\int \int_D 5 - x^2 \sin x + y^3 \cos y \, dA$

(c) $\int \int_D (1 - \sqrt{x^2 + y^2}) \, dA$.

2. Evaluate $\int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2} \, dx \, dy$.

3. Use double integrals to find the volume of the solid that lies inside the ellipsoid $4x^2 + 4y^2 + z^2 = 8$ and above the paraboloid $z = 2x^2 + 2y^2$.

4. Use a double integral to evaluate the area enclosed by one leaf of the four-leaved rose $r = \cos 2\theta$.

5. Find the centroid of the rectangular lamina $\{ -1 \leq x \leq 1, -1 \leq y \leq 1 \}$ with density $\delta(x, y) = 2y + x^2 + 1$. Can you exploit symmetry in any way?

6. Find the mass and centroid of the region bounded by $y = 0, x = -1, x = 1,$ and $y = e^{-x^2}$, with density $\delta(x, y) = |xy|$.

7. Consider a disc $D$ with density $\delta(x, y) = \frac{1}{\sqrt{x^2+y^2}}$, centred at the origin and of radius $a$.

(a) Find the moments of inertia $I_x, I_y, I_0$.

(b) If it is rotating at $4\text{rad/sec}$, what is $KE_{\text{ROT}}$?

(c) Where should I place a very small but very dense weight of the same mass as the disc, so that when rotated about the $z$-axis, it will have the same kinetic energy as the rotating disc?

8. Find the mass and centroid of the region inside the circle $r = 2\sin \theta$ and outside the circle $r = 2$ with density $\delta(x, y) = y$.

9. Find the polar moment of inertia of the right-hand loop of the lemniscate $r^2 = \cos 2\theta$, with density $\delta(x, y) = r^2$. (See Figure 13.5.21).
10. A uniform rectangular plate with base length \(a\), height \(b\), and mass \(m\) is centred the origin. Show that its polar moment of inertia is \(I_0 = \frac{1}{12}m(a^2 + b^2)\).

11. What region \(T\) in \(\mathbb{R}^3\) maximizes the volume of the integral
\[
\iiint_T (1 - x^2 - y^2 - z^2) \, dV?
\]

12. Describe the region of integration for the following integral:
\[
\int_{-1}^{2} \int_{-\sqrt{4 - x^2}}^{\sqrt{4 - x^2}} \int_{-\sqrt{4 - x^2 - y^2}}^{\sqrt{4 - x^2 - y^2}} \, dV
\]

**Warm-Up for next time**

1. Compute the value of the triple integral
\[
\iiint_T x^2 \, dV,
\]
where \(T\) is the tetrahedron bounded by the coordinate planes and the first octant part of the plane with equation \(x + y + z = 1\).