Math 220 AD9 Spring 2009 Worksheet 6

1. Find $\lim_{x\to\infty} f(x)$ if, for all $x > 1$,

$$\frac{5\sqrt{x}}{\sqrt{x-1}} < f(x) < \frac{10e^x - 21}{2e^x}.$$

2. Evaluate the following limits (and justify your answers carefully):

$$\lim_{x\to 0} \sqrt{x}\cos^2\frac{1}{x}, \qquad \lim_{x\to 0} x^2 \sec\frac{1}{x}, \qquad \lim_{x\to 0} xe^{\sin\frac{1}{x}}.$$

3. Given that $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, try to find these other limits by performing any trigonometric tricks you can, or by judiciously changing variables.

   (a) $\displaystyle\lim_{\theta\to 0}\frac{\theta}{\sin(-\theta)}$

   (b) $\displaystyle\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta}$

   (c) $\displaystyle\lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta^2}$

   (d) $\displaystyle\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}$

   (e) $\displaystyle\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$ (hint: look for something to multiply the top and bottom both by to get it more like c)

   (f) $\displaystyle\lim_{\theta\to 0}\frac{3\theta^2}{\sin(-2\theta^2)}.$

   (g) $\displaystyle\lim_{\theta\to 0}\frac{\tan 5x}{\tan -2x}.$

4. Suppose a state's income tax code states that tax liability is 12% on the first \$20,000 of taxable earnings and 16% on the remainder. Write down a piece-wise defined function $T(x)$ that captures this situation. Check that $\lim_{x\to 0^+} T(x) = 0$ and $\lim_{x\to 20{,}000} T(x)$ exists. Why is it important for these limits to exist?

5. Find constants a and b that make the function $f(x)$ continuous.

$$f(x) = \begin{cases} \frac{2\sin x}{x} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ b\cos 3x & \text{if } x > 0 \end{cases}$$

6. There are three parts to the definition of continuity. What are they? For each condition, find an example of a function which breaks that condition.

7. You know the following statement is true:
"If $f(x)$ is a polynomial, then $f(x)$ is continuous."
Which of the following is also true?
(a) If $f(x)$ is not continuous, then it is not a polynomial.
(b) If $f(x)$ is continuous, then it is a polynomial.
(c) If $f(x)$ is not a polynomial, then it is not continuous.

8. Explain the Intermediate Value Theorem in your own words. Use the Intermediate Value Theorem to show $\sqrt[4]{5}$ exists. How could you use the IVT and your calculator to estimate $\sqrt[4]{5}$?

9. Let $g(x) = x^4 - x^3 + x^2 + x - 1$. What are $g(0)$ and $g(1)$? What does this tell you about zeroes of $g(x)$? Explain.

10. True or False: you were once exactly three feet tall. Explain.

11. True or False: at some time since your birth, your weight in pounds exactly equalled your height in inches. Explain.

12. At halftime in the basketball game, your team had 32 points. True or False: at some time your team had exactly 25 points.

13. Use the intermediate value theorem to prove that every polynomial of odd degree has at least one real root.

14. One morning at 6 am, a monk began climbing a tall mountain, which happened to only have one path to the top. He ascended the path at his leisure, taking some stops along the way. He reached the top at 8 pm.

    The next morning at 6 am, the monk descended the mountain along the same path. He took several breaks along the way, and reached the bottom at 8 pm.

    The amazing result: there is some spot on the path that the monk occupied at precisely the same time of day for both trips. Why is this?

15. Q.54, p.108. What do we learn from this question?

16. Determine the intervals on which the following functions are continuous.

$$f(x) = \log(4 - x^2), \qquad g(x) = \frac{1}{1 - x^2}, \qquad h(x) = \frac{1}{1 + x^2}, \qquad k(x) = \frac{2 - x}{x^2 - 4}.$$

Preparation for next time

Read Section 1.5. Do problem 8, p. 118.

Quiz on Friday: Sections 1.2-1.4.

17. Find $\lim_{x\to 0} e^{-\cot x}$.

18. Find $\lim_{x\to 1} \arctan\left(\dfrac{x}{x^2-2x+1}\right)$.

19. Find $\lim_{x\to -\infty} e^{-x^2}$.

20. Evaluate these limits:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}, \qquad \lim_{x\to 0}\,(1+x)^{\frac{1}{x}}.$$

21. True or False: $\lim_{x\to a} f(x) = L$ means that if x_1 is closer to a than x_2 is, then $f(x_1)$ will be closer to L than $f(x_2)$ is. Be prepared to justify your answer with an argument or counterexample.

22. Suppose that $f(x)$ is continuous at $x = 0$. Prove that $\lim_{x\to 0} xf(x) = 0$.
The converse is not true. (What is the converse?) Show this by finding a counterexample: a function f such that $\lim_{x\to 0} xf(x) = 0$ and $f(x)$ is not continuous at $x = 0$.