Math 220 AD9 Spring 2009 Worksheet 21

1. How could you justify using the following (often used) approximation for "small" x?

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x$$

   How do the approximation and the function's actual value compare for $x = 0.01$, $x = 0.1$ and $x = 1$?

2. Consider the function $f(x) = x^3 - 4.41x$. We want to find its zeroes. Note that $f(2)$ is (reasonably) close to zero. What is the tangent line to $f(x)$ when $x = 2$? Where does this tangent line intersect the x-axis? Sketch a graph that shows all of the above. What would you expect the value of $f(x)$ to be at the point you have just found? Why?

3. What is the equation of the tangent line to $f(x)$ when $x = x_0$? Where does this tangent line intersect the x-axis? What is the idea behind the Newton–Raphson method?

4. Use Newton's Method with $x_0 = -1$ to estimate the location of a root to the equation $x^3 + 4x^2 - x - 1 = 0$. Compute x_1 and x_2 by hand. Afterwards, continue with your calculator to find the root correct to 5 decimal places. How do you know how many iterations you have to perform/ how long you have to continue?

5. What is an easy to differentiate function which is 0 when $x = \sqrt{2}$? Use this (and your calculator) to estimate $\sqrt{2}$ (correct to 6 decimal places).

   If you did not know about the Newton–Raphson method or linear approximation, how would you accurately estimate the value of $\sqrt{2}$? (Yes, you can use a calculator but only after you tell the manufacturer how to program it.) How would you accurately estimate the thousands of necessary calculations that keep this building standing, the electricity running (most of the time), and airplanes in the air?

6. What is an easy to differentiate function which is 0 when $x = e$? Use this and your calculator to estimate e (to 6 decimal places). How many correct decimal places do you have after each iteration?

7. Use Newton's method to find the zero of the function

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0. \end{cases}$$

   What is going on here?

8. Try using Newton's Method to find a root of the equation $x^3 - 2x + 2 = 0$, with initial value $x_0 = 0$. Explain your results.

9. We could (informally) describe the limit $\lim_{x\to\infty} x^x$ as ∞^∞. It's pretty clear what this limit should be (what should it be?).

   Sometimes it is not so easy to say what the limit should be. In this case, we call this an indeterminate form. An example of an indeterminate form is 1^∞.

   For example, what about $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$?

   Give a reason why you might expect this limit to be 1.

   Give a reason why you might expect this limit to be ∞.

   Which of the following are indeterminate forms? For those that are not, give the value of the form.

$$\infty^0 \quad 0^\infty \quad 0^0 \quad \infty-\infty \quad \infty+\infty$$
$$\infty*\infty \quad \infty*(-\infty) \quad 0*\infty \quad \frac{0}{0} \quad \frac{\infty}{\infty} \quad \infty^\infty$$

   What is $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$?

   For those forms that are indeterminate, see if you can find examples of where this limit is ∞ and examples where it is 0 or 1.

10. Consider the following limits:

$$\lim_{x\to\infty} \frac{x}{x^2}, \qquad \lim_{x\to\infty} \frac{x^5}{x^3}, \qquad \lim_{x\to\infty} \frac{2x+3}{5x+1}.$$

    All of these are of the form $\frac{\infty}{\infty}$. Do not cancel out the x's and do not use any sophisticated rules we have not met yet. Informally explain what you would expect the limit to be by comparing the numerator to the denominator. What's the important issue here? How about $\lim_{x\to\infty} \frac{\sqrt{x}}{\ln x}$?

Preparation for next time

Read Section 3.2. You will be asked about the material from Section 3.2 at the beginning of the next merit section.

Potential questions (for this and future sections) include: What is the main point or main idea of this section? This section answers what question? What will be able to do after this section that we could not do before? How would you solve this simple example of a problem from this section?