Math 220 AD9 Spring 2009 Worksheet 16

Note: Calculus always builds on what we have learned before. It is very important to not get left behind. Notice how the examples below use not only the new rule for differentiating exponentials but also the product, chain, quotient, sum and power rules and the rules for differentiating trigonometric functions; this is typical of what you should expect in future.

1. Sketch the graph of $f(x) = e^x$. Use this to sketch the graph of the derivative $f'(x)$. What is the derivative of e^x?

   Use the definition of the derivative to try and find the derivative of $y = a^x$. Looking at your answer, why is it more convenient/natural to use the base e for exponentials?

   We can write $10 = e^{\log 10}$. Use this fact and the chain rule to find the derivative of $y = 10^x$. Similarly, what is the derivative of $y = a^x$?

2. Find the derivatives of the following functions

$$\begin{array}{llllll} (a) & x^2 e^{5x} & (b) & \tan x + 2^x + 3^x & (c) & e^{x^3+5x} \\ (e) & 4^{-x^2} & (f) & e^{\sin 2x} & (g) & \dfrac{e^{4x}}{x} \end{array}$$

3. Sketch the graph of $f(x) = \log |x|$. Use this to sketch the graph of the derivative $f'(x)$. What is the derivative of $\log |x|$?

4. Find the derivatives of the following functions.

$$\begin{array}{llllll} (a) & \ln 2 + \ln x & (b) & \ln 2x & (c) & x^3 \ln(x^2+1) \\ (d) & \sin(\ln(\cos x^3)) & (e) & \ln(x + e^x) & (f) & e^{4x} \ln(5x+2) \end{array}$$

5. What functions does the power rule for differentiation apply to?
   What functions does the exponential rule for differentiation apply to?
   Do either of these apply to the function $y = x^x$?

6. What is the derivative of $f(t) = \ln g(t)$? Solve this equation for $g'(t)$. Can you use this to differentiate $y = x^x$?

   Note: Sometimes you can make your life a lot easier by taking logarithms before you attempt limits and derivatives. Notice that if $f(x) = g(x)^{h(x)}$ then we can do the following:

$$\begin{aligned} \frac{d}{dx} \log f(x) &= \frac{d}{dx}[\log g(x)^{h(x)}] \\ \frac{f'(x)}{f(x)} &= \frac{d}{dx}[h(x) \log g(x)] \\ f'(x) &= f(x) \frac{d}{dx}[h(x) \log g(x)], \end{aligned}$$

7. Differentiate the following functions

$$\begin{array}{llll} (a) & y = x^x \text{ for } x > 0 & (b) & y = (\sin x)^x \text{ for } \pi > x > 0 \\ (c) & y = x^{\sin x} \text{ for } x > 0 & (d) & y = x^{\sqrt{x}+\ln x} \text{ for } x > 0. \end{array}$$

(Hint: your first step should be to take the natural log of both sides of the equation.) Why are there restrictions on what x we consider? What types of functions do you differentiate using logarithmic differentiation?

8. Find the equation of the tangent line to $f(x) = 2e^{x-1}$ at $x = 1$.

9. Find the derivative of $y = \sqrt{x+1}\sqrt[3]{x+2}\sqrt[4]{x+3}$. You can use the product and chain rules, but I think that you will find it much easier if you use logarithmic differentiation.

   Note: There is another trick to compute derivatives that you may find useful. Instead of taking the log of both sides, you can rewrite a function using exponentials and logs. For example, $f(x) = x^x = e^{\log x^x} = e^{x \log x}$. The derivative of the last form can be evaluated using rules you already know!

10. Find the derivative of $f(x) = x^x$ using the above method.

11. Find the derivative of $f(x) = x^{1/x}$ using the new method.

12. A bacterial population starts at 200 and triples every day. Find a formula for the population after t days and find the percentage rate of change in population.

13. The concentration of a certain chemical after t seconds of an autocatalytic reaction is given by $x(t) = \dfrac{6}{2e^{-8t}+1}$. Show that $x'(t) > 0$ and use this information to determine that the concentration of the chemical never exceeds 6.

Preparation for next time

Read Section 2.8. Do problem 3, p. 224.