Math 220 AL1, Spring 2009, Practice Exam 4

Important: This is merely a study aid. I have not seen the exam and will not be writing the exam. There will be things on the exam which are not on this practice exam.

1. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min in such a way that it forms a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

   In related rates problems, it usually helps to start by sketching the situation and writing down the relationships between the variables.

   Volume of a cone with height h and radius r is $\frac{1}{3}\pi r^2 h$.
   Here $2r$ = base diameter = height, so $r = \frac{h}{2}$.
   Volume at time t, $V(t) = \frac{1}{3}\pi\frac{h(t)^3}{4} = \frac{1}{12}\pi h(t)^3$.
   Differentiating both sides with respect to t,

   $$V'(t) = \frac{1}{4}\pi h(t)^2 h'(t)$$

   We know that $V'(t) = 30$ and want to know $h'(t)$ when $h(t) = 10$. Fill in $V'(t) = 30, h(t) = 10$ and then solve for $h'(t)$.

   $$30 = \frac{1}{4}\pi 100 h'(t), \qquad \frac{1.2}{\pi} = h'(t).$$

2. What is the average value of the function $f(x) = \dfrac{x}{x+2}$ on the interval $[0, 2]$?

   The average value of a function $f(x)$ on an interval $[a, b]$ is given by $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

   $$\frac{1}{2}\int_0^2 \frac{x}{x+2}\,dx$$

   How do we solve this? If we make the u-substitution $u = x + 2$, then $du = dx$. When $x = 0$, $u = 2 + 0 = 2$. When $x = 2, u = 2 + 2 = 4$. We have now dealt with everything except for the x – how do we write this in terms of u? Seeing as $u = x + 2$, we have $u - 2 = x$. This gives us the integral

   $$\begin{aligned}&\frac{1}{2}\int_2^4 \frac{u-2}{u}\,du = \frac{1}{2}\int_2^4 1 - \frac{2}{u}\,du\\ =&\frac{1}{2}\left(u - 2\ln u\right)\Big|_2^4 = \frac{1}{2}\left(4 - 2\ln 4 - 2 + 2\ln 2\right)\\ =&1 - \ln 4 + \ln 2 = 1 - \ln 2\end{aligned}$$

3. Evaluate the following definite integral using Riemann sums.
(0 points for solving it using the fundamental theorem of Calculus!)

$$\int_{-1}^{1} 3x^2 + x \; dx$$

Our first step is to cut the interval into n pieces and find the corresponding Riemann sum. Our second step will be to take the limit (as $n \to \infty$) of our answer from step one.

Divide the interval $[-1, 1]$ into n equal subintervals each of length $\frac{1-(-1)}{n} = \frac{2}{n}$ – this is the width of the rectangles. Then the subintervals will be $[-1 + \frac{2(i-1)}{n}, -1 + \frac{2i}{n}]$. I will use the left endpoints as my evaluation points – I will use $f(-1 + \frac{2i}{n})$ as the height of the i^{th} rectangle. So my Riemann sum is

$$\begin{aligned}
&\sum_{i=1}^{n} \left(3\left(-1 + \frac{2i}{n}\right)^2 + \left(-1 + \frac{2i}{n}\right)\right) \frac{2}{n} \\
=&\sum_{i=1}^{n} \left(3\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - 1 \; + \frac{2i}{n}\right) \frac{2}{n} \\
=&\sum_{i=1}^{n} \left(2 - \frac{10i}{n} + \frac{12i^2}{n^2}\right) \frac{2}{n} \\
=&\sum_{i=1}^{n} \frac{4}{n} - \frac{20}{n^2} \sum_{i=1}^{n} i + \frac{24}{n^3} \sum_{i=1}^{n} i^2 \\
=&4 - \frac{20}{n^2} \frac{n(n+1)}{2} + \frac{24}{n^3} \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

We now take the limit as $n \to \infty$ to obtain the definite integral.

$$\begin{aligned}
&\lim_{n\to\infty} 4 - \frac{20}{n^2} \frac{n(n+1)}{2} + \frac{24}{n^3} \frac{n(n+1)(2n+1)}{6} \\
=&\lim_{n\to\infty} 4 - \frac{20}{n^2} \frac{n^2+n}{2} + \frac{24}{n^3} \frac{2n^3+3n^2+n}{6} \\
=&\lim_{n\to\infty} 4 - 10\left(1 + \frac{1}{n}\right) + 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \\
=&4 - 10 + 8 = 2
\end{aligned}$$

Our final answer is 2 – answers to definite integrals like this (where the limits are numbers) should always be numbers and not functions.

4. If the velocity of a particle at time t is given by $v(t) = t^3 + t^2 - 2t$, what is the total distance travelled between $t = 0$ and $t = 2$?

   If $s(t)$ is the position (or displacement) at time t, then $v(t) = s'(t)$. How do we find s if we know v? We integrate.

   We are asked for the distance travelled and not just the position or displacement. We have to distinguish between travelling in the "positive" direction and travelling in the "negative" direction. We break up the integral according to where $v(t)$ is positive or negative.

$$v(t) = t^3 + t^2 - 2t = t(t^2 + t - 2) = t(t + 2)(t - 1),$$

   so $v(t)$ is negative on $(-\infty, -2)$ and $(0, 1)$ (and positive everywhere else).

$$\int_0^1 t^3 + t^2 - 2t\ dt = \frac{1}{4}t^4 + \frac{1}{3}t^3 - t^2\Big|_0^1 = \frac{1}{4} + \frac{1}{3} - 1 = -\frac{5}{12}\ \text{(Negative!!)}$$

$$\int_1^2 t^3 + t^2 - 2t\ dt = \frac{1}{4}t^4 + \frac{1}{3}t^3 - t^2\Big|_1^2 = \frac{1}{4}16 + \frac{1}{3}8 - 4 - (\frac{1}{4} + \frac{1}{3} - 1) = \frac{37}{12}$$

   So our final answer is

$$\frac{37}{12} + \frac{5}{12} = \frac{42}{12} = \frac{7}{2}$$

5. Find the equation of the tangent line to the curve

$$y = \int_x^5 e^{t^2-25}\ dt,$$

   at $x = 5$.

   Remind yourself what you would do if you were asked to find the tangent line to the curve $y = x^2 + 1$ at $x = 5$. We differentiate and then evaluate the derivative at $x = 5$ to find the slope of the tangent line. We fill in $x = 5$ into the original function to find the y-value of the point that the tangent line passes through. We follow the same procedure here.

   Our equation for this curve is not in exactly the same shape as the functions we have seen in the Fundemental Theorem of Calculus. We rewrite it as $y = -\int_5^x e^{t^2-25}\ dt$.

   By the Fundamental Theorem of Calculus, the derivative is $\frac{dy}{dx} = -e^{x^2-25}$.
   At $x = 5$, the derivative is $-e^{25-25} = -e^0 = -1$.

   What is the point the tangent line passes through? Fill in $x = 5$ to get $\int_5^5 e^{t^2-25}\ dt = 0$.
   What is the area under the curve and above the x-axis between $x = 5$ and $x = 5$? Zero.

   So our tangent line is

$$y - 0 = -1(x - 5),\ \text{or}\ y = 5 - x.$$

6. Find the following integrals:

(a) $\int_{-1}^{1} x^3(4x^3+x^8)\ dx$

$$\begin{aligned} &= \int_{-1}^{1} 4x^6 + x^{11}\ dx \\ &= \left(\frac{4}{7}x^7 + \frac{1}{12}x^{12}\right)\Big|_{-1}^{1} \\ &= \frac{4}{7} + \frac{1}{12} - \left(\frac{-4}{7} + \frac{1}{12}\right) = \frac{8}{7} \end{aligned}$$

(b) $\int x^4(\sin x^5)^7 \cos x^5\ dx$

$$u = \sin x^5 \qquad du = \left(\cos x^5\right) 5x^4\ dx$$

$$\frac{1}{5}du = x^4 \cos x^5\ dx$$

$$\int \frac{1}{5}u^7\ du = \frac{1}{40}u^8 + c$$

Our answer should be a function of x, so we fill back in for x to get

$$\frac{1}{40}\left(\sin x^5\right)^8 + c.$$

(c) $\int \frac{1}{x(\ln x)^4}\ dx$

$$u = \ln x \qquad du = \frac{1}{x}\ dx$$

$$\int \frac{1}{u^4}\ du = \int u^{-4}\ du = \frac{1}{-3}u^{-3} + c$$

and our final answer is

$$\frac{1}{-3}\frac{1}{(\ln x)^3} + c.$$

(d) $\int_{1}^{2} xe^{2x^2}\ dx$

$$u = 2x^2 \qquad du = 4x\ dx, \text{ so } \frac{1}{4}du = x\ dx.$$

Change the limits of integration. When $x = 1$, $u = 2(1)^2 = 2$. When $x = 2$, $u = 2(2)^2 = 8$.

$$\int_{2}^{8} \frac{1}{4}e^u\ du = \frac{1}{4}\ e^u|_2^8 = \frac{e^8 - e^2}{4}.$$

7. Evaluate the integral

$$\int_0^{\pi/12} \tan 3x \; dx$$

We can rewrite this as

$$\int_0^{\pi/12} \frac{\sin 3x}{\cos 3x} \; dx$$

We now make the substitution $u = \cos 3x$. Next we find $du = -3\sin 3x \; dx$ and so $-\frac{1}{3}du = \sin 3x \; dx$. Remember to change the limits of integration. When $x = 0$, $u = \cos 3(0) = \cos 0 = 1$. When $x = \pi/12$, $u = \cos 3(\pi/12) = \cos \pi/4 = \frac{1}{\sqrt{2}}$. We have now rewritten the above integral as

$$\int_1^{1/\sqrt{2}} -\frac{1}{3}\frac{1}{u} \; du.$$

Now we integrate, to get

$$-\frac{1}{3} \ln u|_1^{1/\sqrt{2}} = -\frac{1}{3}\left(\ln 1/\sqrt{2} - \ln 1\right) = -\frac{1}{3} \ln \frac{1}{\sqrt{2}}.$$

8. Use Simpson's Rule with $n = 4$ to estimate the integral.

$$\int_1^3 x^2 \; dx$$

How does your estimate compare with the actual value of the integral? Why?

We break the interval $[1, 3]$ into four intervals with the (five) endpoints $1, \frac{3}{2}, 2, \frac{5}{2}, 3$. We now fill into the formula:

$$\frac{3-1}{3(4)}\left((1)^1 + 4(\frac{3}{2})^2 + 2(2)^2 + 4(\frac{5}{2})^2 + 3^2\right) = \frac{1}{6}\,(1 + 9 + 8 + 25 + 9) = \frac{52}{6} = \frac{26}{3}.$$

This estimate is equal to the actual value of the integral. Simpson's Rule works by replacing the function $f(x)$ by finding the area under the parabolas that go through the same points as $f(x)$ at the endpoints of our subintervals. Seeing as we started out with a parabola, Simpson's Rule will exactly solve the integral.