Math 220 AL1, Spring 2009, Practice Exam 2

Important: This is merely a study aid. I have not seen the exam and will not be writing the exam. There will be things on the exam which are not on this practice exam.

1. Find the derivatives of the following functions

   (a) $(x^2 + 4x - 2)\tan x$

   $$(x^2 + 4x - 2)\sec^2 x + (2x + 4)\tan x \quad \text{Product Rule}$$

   (b) $\dfrac{2x - 4}{x^3 + \sqrt{x}}$

   $$\frac{(x^3 + \sqrt{x})(2) - (2x - 4)(3x^2 + \frac{1}{2}x^{-1/2})}{(x^3 + \sqrt{x})^2} \quad \text{Quotient Rule}$$

   (c) $\sin\left(e^{\cos(5x - 4)}\right)$

   $$\cos\left(e^{\cos(5x - 4)}\right) e^{\cos(5x - 4)}(-\sin(5x - 4))5 \quad \text{Chain Rule}$$

   (d) $x^\pi \ln(3x^2 + 1)$

   $$x^\pi \frac{1}{3x^2 + 1} 6x + \pi x^{\pi - 1} \ln(3x^2 + 1) \quad \text{Product and Chain Rules}$$

2. Find the equation of the tangent line to

$$y = e^{x^2-1}$$

at $x = 1$.

$\frac{dy}{dx} = (2x)e^{x^2-1}$ – Chain rule.

The derivative is not the tangent line. When I fill in $x = 1$ into the derivative, I find the slope of the tangent line at $x = 1$. So the slope I am looking for is

$2e^{1-1} = 2e^0 = 2$

and it goes through the point $(1, e^{1-1}) = (1, e^0) = (1, 1)$.

The equation of the tangent line is

$y = 2(x - 1) + 1$.

3. Find $f^{(2)}(\frac{1}{2})$ if

$$f(x) = \arctan 2x$$

This notation means I differentiate twice and then I fill in $x = 1/2$ (but only *after* differentiating twice).

$f^{(1)}(x) = \frac{1}{1+(2x)^2}2 = \frac{2}{1+4x^2} = 2(1 + 4x^2)^{-1}$

$f^{(2)}(x) = (2)(-1)(1 + 4x^2)^{-2}(8x)$

$f^{(2)}(\frac{1}{2}) = 2(-1)(1 + 1)^{-2}(4) = -2$

4. If the height of a rocket at time t is given by

$$s(t) = -t^3 + 6t^2 + 15t + 10,$$

(a) find the velocity and acceleration of the particle at time t.

$$s'(t) = -3t^2 + 12t + 15$$
$$s''(t) = -6t + 12$$

(b) When does the rocket reach its greatest height?

It reaches its greatest height when the velocity is (momentarily) 0. So let's solve $s'(t) = 0$ for t.

$$\begin{aligned} -3t^2 + 12t + 15 &= 0 \\ t^2 - 4t - 5 &= 0 \\ (t-5)(t+1) &= 0 \end{aligned}$$

The two options are $t = 5$ and $t = -1$. Seeing as the rocket was (presumably) fired at time $t = 0$, the only one of these options that make sense is $t = 5$.

5. Find the derivative of

$$f(x) = (\cos x)^{\cos x},$$

for $|x| < \pi/2$.

We use logarithmic differentiation (as this is a variable to the power of a variable).

See my webpage for a few more comments on logarithmic differentiation (and implicit differentiation).

$$\begin{aligned}
\ln f(x) &= \cos x \ln \cos x \\
\frac{1}{f(x)} f'(x) &= \cos x \frac{1}{\cos x}(-\sin x) + (-\sin x) \ln \cos x \\
f'(x) &= (\cos x)^{\cos x} \left(-\sin x + (-\sin x) \ln \cos x\right)
\end{aligned}$$

Remember to multiply across by $f(x)$ and fill in for $f(x) = (\cos x)^{\cos x}$.

6. Find an equation of the tangent line to

$$x^3y^2 = -3xy$$

at the point $(-1, -3)$.

We use implicit differentiation. Differentiate both sides of the equation and then solve for $\frac{dy}{dx}$.

$$\begin{aligned} 3x^2y^2 + x^3(2y)\frac{dy}{dx} =& -3y - 3x\frac{dy}{dx} \\ 3x^2y^2 + 3y =& (-3x - 2x^3y)\frac{dy}{dx} \\ \frac{3x^2y^2 + 3y}{-3x - 2x^3y} =& \frac{dy}{dx} \end{aligned}$$

We want the slope of the tangent line at $(-1, -3)$, so we fill in $x = -1$ and $y = -3$.

$$\frac{3(-1)^2(-3)^2 + 3(-3)}{-3(-1) - 2(-1)^3(-3)} = \frac{27 - 9}{3 - 6} = \frac{18}{-3} = -6$$

The slope is -6 and it goes through $(-1, -3)$, so the equation of the tangent line is

$$y + 3 = -6(x + 1).$$

7. What does the mean value theorem tell you about the location of the zeros of $f'(x)$ if

$$f(x) = (x-3)(x-5)(x-10)(x-15)(x-18)?$$

Explain your answer.

There are zeros of $f'(x)$ between 3 and 5, between 5 and 10, between 10 and 15, between 15 and 18.

$f(3) = 0$ and $f(5) = 0$ – the Mean Value Theorem (or the special case known as Rolle's Theorem) says that there exists a number c, $3 < c < 5$, such that $f'(c) = 0$.

(Rolle's Theorem says exactly this. The Mean Value Theorem says that there is a c, $3 < c < 5$, with

$$f'(c) = \frac{f(5) - f(3)}{5-3} = \frac{0}{2} = 0.)$$

Either way, this c will be a zero of $f(x)$. Similarly, there will be zeros in the other ranges mentioned above.