Logarithmic Differentiation

- When do you use logarithmic differentiation?

  You use logarithmic differentiation when you have expressions of the form $y = f(x)^{g(x)}$, a variable to the power of a variable. The power rule and the exponential rule do not apply here.

- Why do you use logarithmic differentiation?

  The advantage is that you can now write this as $\ln y = g(x) \ln f(x)$ – this can now be differentiated using the chain rule on the left and the product and chain rules on the right.

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} =& g(x)\frac{1}{f(x)}f'(x) + g'(x)\ln f(x)\\ \frac{dy}{dx} =& y\left(g(x)\frac{1}{f(x)}f'(x) + g'(x)\ln f(x)\right)\\ \frac{dy}{dx} =& f(x)^{g(x)}\left(g(x)\frac{1}{f(x)}f'(x) + g'(x)\ln f(x)\right)\end{aligned}$$

  You want to find $\frac{dy}{dx}$ (not $\frac{1}{y}\frac{dy}{dx}$), so multiply across by $y = f(x)^{g(x)}$. Your answer should feature only x's. You know what y is equal to; use that information.

- Look at this example $y = x^{e^x}$.

$$\begin{aligned}\ln y =& e^x \ln x\\ \frac{1}{y}\frac{dy}{dx} =& e^x\frac{1}{x} + e^x \ln x\\ \frac{dy}{dx} =& x^{e^x}\left(e^x\frac{1}{x} + e^x \ln x\right)\end{aligned}$$

- You could also use logarithmic differentiation to simplify some other expressions:

$$\begin{aligned}y =& \sqrt{x+1}\sqrt[3]{x+2}\sqrt[4]{x+3}\\ \log y =& \frac{1}{2}\log(x+1) + \frac{1}{3}\log(x+2) + \frac{1}{4}\log(x+3),\end{aligned}$$

  where the second expression is easier to differentiate than the first.

Implicit Differentiation

An expression like $y = x^2 + 3$ defines the function y *explicitly* – it says openly and directly how to find y if you know x. If I have an expression like $x = \arctan(x + y)$, the connection between x and y is still there but only *implicitly*, the link is hidden and the xs and ys are tangled together.

- When do you use implicit differentiation?

  You use implicit differentiation when the function y is only implicitly defined. When it is difficult (or impossible) to solve and get $y = \dots$, we use implicit differentiation.

- How do you use implicit differentiation?

  You differentiate both sides of the equation and then solve for $\frac{dy}{dx}$. It is important to remember that y is a function of x and you have to use the product, quotient and chain rules just as you would for any other function of x.

- An example:

$$\begin{aligned} xy &= x^2 + y^2 \\ 1 \cdot y + x\frac{dy}{dx} &= 2x + 2y\frac{dy}{dx} \\ y - 2x &= (2y - x)\frac{dy}{dx} \\ \frac{y - 2x}{2y - x} &= \frac{dy}{dx} \end{aligned}$$

  xy is a product of two functions of x so we use the product rule. The derivative of y is $\frac{dy}{dx}$, which you can also write y' (but make sure not to confuse this with y^1, which has a very different meaning). On the right, we use the chain rule to differentiate y^2. Seeing as we do not have a neat way of writing y in terms of x, it is fine if your answer includes ys.

- Finding $\frac{d^2y}{dx^2}$.

  First we find $\frac{dy}{dx}$ and we differentiate again. Remember that y is still a function of x and must be treated in the same way as above. Finally, our answer for $\frac{d^2y}{dx^2}$ should contain only xs and ys and not $\frac{dy}{dx}$. Anywhere we have a $\frac{dy}{dx}$, we should fill in our answer from above.

  Returning to the example above, we use the quotient rule to get

$$\begin{aligned} \frac{(2y - x)(\frac{dy}{dx} - 2) - (y - 2x)(2\frac{dy}{dx} - 1)}{(2y - x)^2} &= \frac{d^2y}{dx^2} \\ \frac{(2y - x)(\frac{y-2x}{2y-x} - 2) - (y - 2x)(2\frac{y-2x}{2y-x} - 1)}{(2y - x)^2} &= \frac{d^2y}{dx^2}. \end{aligned}$$