Here are some comments on the questions I graded on exam 1 and on a few recurring mistakes I noticed.

Composition of functions

What is $(f \circ g \circ h)(x)$?

$$f(x) = \frac{1}{x+1}, \quad g(x) = x + \ln x, \quad h(x) = x^2 - 1.$$

You start with whatever is furthest to the right - it is closest to the x so you apply that function to x first. So we start with

$$h(x) = x^2 - 1.$$

Next, what is $(g \circ h)(x)$? This is $g(h(x))$ - every place there is an x in the function $g(x)$, we replace it by $h(x) = x^2 - 1$.

$$(g \circ h)(x) = g(h(x)) = (x^2 - 1) + \ln (x^2 - 1).$$

Finally, what is $f \circ g \circ h$? Everywhere there is an x in $f(x)$, we replace it by our answer from above:

$$(f \circ g \circ h)(x) = f(g(h(x)) = \frac{1}{x^2 - 1 + \ln (x^2 - 1) + 1}.$$

In all problems, you are expected to know and use the basic rules of logs, powers and trigonometric functions. You cannot simplify the bottom further here. In particular, $\ln A + \ln B \neq \ln(A + B)$ (NOT EQUAL). The rules of logs tell us $\ln A + \ln B = \ln(A \cdot B)$, which does not help here. Also $A + \ln B \neq \ln(A + B)$ (NOT EQUAL). Some stuff is inside the log and some stuff here is outside the log and you cannot change that.

Domains of functions

What is the domain of

$$f(x) = \frac{\sqrt{x+2}}{\ln (x^2 + 9x) + x^2 + 9x + 7}?$$

Another way of phrasing this question is: for what x does the above expression make sense? It is easy to write something down which looks good at first glance but which is not a real number (for example, $\sqrt{-5}$). We need *all* parts of this to make sense. The following example shows three ways a function could fail to be defined: you cannot take the square root of a negative number, you cannot take the log of a negative number or of 0, and you cannot divide by 0.

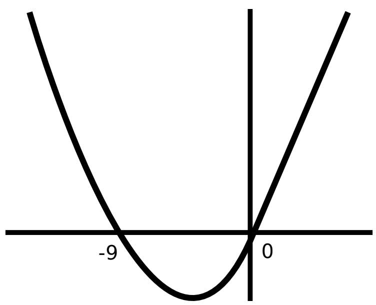
There are three main parts to this function:

- $\sqrt{x+2}$: When does this make sense? We need the input of the squareroot function to be greater than or equal to 0. We need $x+2 \geq 0$ or $x \geq -2$.

  The restriction here is *not* $x \geq 0$. Yes, we can only take the square roots of nonnegative numbers but the input into the squareroot is not x; instead our input is $x+2$.

- $\ln(x^2+9x)$: When does this make sense? We can only take the natural log of positive numbers. The domain of ln is $(0,\infty)$. So the restriction here is $x^2+9x > 0$. Section 0.1 includes inequalities like this (and, yes, you will probably see inequalities like this again in this course).

  $$x^2 + 9x = x(x+9) > 0$$

  By looking at a rough sketch of this quadratic, we see that it is positive when

  $$x < -9 \text{ or } x > 0, \text{ which we can write as } (-\infty,-9) \cup (0,\infty).$$

  Yes, x^2 is always positive, but that does not mean that x^2+9x is always positive (as we can see from the graph above). The x^2 is positive but the $9x$ is sometimes negative and so the two added together will sometimes be negative.

- Dividing by the denominator. Let's first look at a simpler and more typical example. What is the domain of

  $$g(x) = \frac{1}{x^2-5x+6}?$$

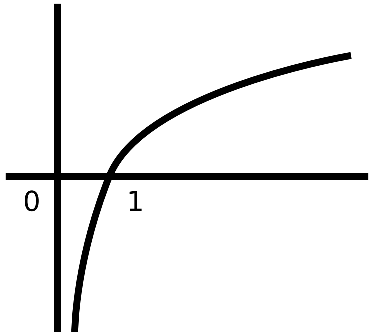
  Well, we are not allowed to divide by zero. Where is the denominator zero?

  $$g(x) = \frac{1}{(x-3)(x-2)},$$

  so the domain of $g(x)$ is all real numbers, except for $x=2$ and $x=3$.

  Now let's return to the example above. Here (as in the question on the exam), it would be very, very tricky to say exactly what x makes the bottom zero, so it is enough to just say "not those x for which $\ln(x^2+9x) + x^2 + 9x + 7 = 0$".

  Yes, it is possible for this to happen. Some people on the exam incorrectly stated that ln is always positive - this is incorrect. Look at its graph:



It only accepts positive *inputs*. Its output can be negative - in fact, $\ln x < 0$ if $x < 1$ and it has a vertical asymptote heading off to $-\infty$ at $x = 0$.

We can see that the denominator must be zero somewhere by applying the intermediate value theorem. Let's call the denominator $f(x)$, so $f(x) = \ln(x^2 + 9x) + x^2 + 9x + 7$. Then $f(1) = \ln(10) + 10 + 7 > 0$. If x is a very small positive number, then $\ln(x^2 + 9x)$ is a very large negative number and so $f(x) \approx$ large negative number $+\ 7 < 0$. It's a continuous function which is positive at $x = 1$ and negative near 0, so somewhere in between 0 and 1, it must cross the x-axis. Let's call this point x_0 - we haven't gone to the bother of finding an exact value for x_0 but we know it exists. (No, you did not have to do this last part on the test, but you should have pointed out that you cannot divide by 0.)

Finally, let's put it altogether. All three parts of the function must make sense, so we must satisfy all three of the restrictions that we found above.

We must have

- $x \in [-2, \infty)$
- $x \in (-\infty, -9) \cup (0, \infty)$
- denominator not equal to 0 (for example, not x_0)

which is

$$x \in (0, \infty) \text{ and the denominator not equal to } 0.$$