Limits of rational functions at infinity

What is the value of the limit

$$\lim_{x\to\infty} \frac{x^3+3x^2+1}{2x^3-7x+5}?$$

Do not "fill in" $x = \infty$. Remember that ∞ is *not* a number. Calculations using ∞ as if it is a number will give you the wrong answer 99% of the time. In particular, you cannot say $\frac{\infty}{\infty} = 1$ or $\frac{\infty}{\infty} = \infty$ - these expressions are meaningless and will not help you get the right answer.

Also, we cannot break this up using the sum and quotient rules for limits. These rules only apply if the limits involved in the sum and quotient exist. That is,

$$\lim_{x\to\infty} \frac{x^3+3x^2+1}{2x^3-7x+5} \neq \frac{\lim_{x\to\infty} x^3+3x^2+1}{\lim_{x\to\infty} 2x^3-7x+5} \text{``}=\text{''} \frac{\infty}{\infty} = ???,$$

because the limits on the right hand side do not exist.

We cannot deal with ∞ directly so we re-write the limit so that no infinities will appear. The most common way of doing this is to divide above and below by the highest power of x to appear in the function.

$$\lim_{x\to\infty} \frac{x^3+3x^2+1}{2x^3-7x+5} = \lim_{x\to\infty} \frac{(x^3)(1+\frac{3}{x}+\frac{1}{x^3})}{(x^3)(2-\frac{7}{x}+\frac{5}{x^3})} = \lim_{x\to\infty} \frac{1+\frac{3}{x}+\frac{1}{x^3}}{2-\frac{7}{x}+\frac{5}{x^3}}$$

We have now rewritten this to avoid ∞ and all the limits on the right-hand side do now exist. We can now use the sum and quotient rules to simplify this expression. For full credit, you need to do at least the amount of work displayed above.

We now have

$$\frac{\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{3}{x} + \lim_{x\to\infty} \frac{1}{x^3}}{\lim_{x\to\infty} 2 - \lim_{x\to\infty} \frac{7}{x} + \lim_{x\to\infty} \frac{5}{x^2}} = \frac{1}{2}.$$

For a 10 point question, you need to provide some explanation; you are not going to receive 10% of the marks for the entire exam for only writing down $\frac{1}{2}$. People who talked about "leading terms" and explained that for large x only the x^3 terms would be significant also received full credit.

Similarly,

$$\lim_{x\to\infty} \frac{4x^3-5x^2+1}{2x^4-9x^3+x} = \lim_{x\to\infty} \frac{\frac{4}{x}+\frac{-5}{x^2}+\frac{1}{x^4}}{2x-\frac{9}{x}+\frac{1}{x^3}} = \cdots = \frac{0}{\lim_{x\to\infty} 2x} = 0,$$

and (dividing by the highest power in the denominator)

$$\lim_{x\to\infty} \frac{3x^5-2x^3+7}{5x^3+4x-1} = \lim_{x\to\infty} \frac{3x^2-2+\frac{7}{x^3}}{5+\frac{4}{x^2}-\frac{1}{x^3}} = \infty,$$

where we justify the last step by saying the numerator tends to ∞ and the denominator tends to 5.