Merit Math 220, Spring 2009, Review Sheet for Final Exam

The usual warnings apply. I am not writing the final exam.

1. Evaluate the following limits. Justify your work.

   (a) $\lim_{x\to 0} \frac{5x}{\sin 3x}$ $= \lim_{x\to 0} \frac{5}{3} \frac{3x}{\sin 3x} = \frac{5}{3} \cdot 1 = \frac{5}{3}$

   $\lim_{x\to 0} \frac{\sin x}{x} = 1.$

   (b) $\lim_{x\to \pi} \frac{\cos x}{2x^2+3}$ $= \frac{\cos \pi}{2\pi^2+3} = \frac{-1}{2\pi^2+3}$

   Continuous function? Just fill in.

   (c) $\lim_{x\to \infty} \frac{2x^3+5x^2+x+1}{1-x^2}$ $= \lim_{x\to\infty} \frac{2x+5+\frac{1}{x}+\frac{1}{x^2}}{-1+\frac{1}{x^2}}$

   Numerator $\to +\infty$, Denominator $\to -1$

   Answer: $-\infty$.

   (d) $\lim_{x\to \infty} \frac{\ln(x^2+3)}{x}$ "Looks like" $\frac{\infty}{\infty}$.

   Use L'Hôpital's Rule

   $= \lim_{x\to\infty} \frac{\frac{1}{x^2+3}(2x)}{1} = \lim_{x\to\infty} \frac{2x}{x^2+3} = \lim_{x\to\infty} \frac{2}{x+\frac{3}{x}} = 0.$

2. Consider the following function

$$f(x) = \begin{cases} x^2 + \frac{3}{x} & x \leq 1 \\ 3 & x = 1 \\ 3x + 1 & x > 1 \end{cases}$$

What is $\lim_{x \to 1} f(x)$?

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3x + 1 = 4$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 + \frac{3}{x} = 1 + \frac{3}{1} = 4.$$

$$\lim_{x \to 1} f(x) = 4$$

Where is $f(x)$ continuous? Justify your answers.

$f(x)$ is continuous everywhere except $x = 0$ and $x = 1$.

There is an asymptote at $x = 0$ and $\lim_{x \to 1} f(x) = 4 \neq f(1) = 3$ so not continuous at 1.

5. Differentiate the following functions.

(a) $\dfrac{\ln(x^2+3)}{x}$ Quotient Rule + Chain Rule

$$\frac{x\,\frac{2x}{x^2+3} - \ln(x^2+3)(1)}{x^2}$$

(b) $(x^2+1)\sin 4x$ Product Rule (& a little chain rule)

$$(x^2+1)(4\cos 4x) + 2x\sin 4x$$

(c) $5x+7+e^{\sqrt{5x+7}}$ Sum + Chain Rules

$$5+0+e^{\sqrt{5x+7}}\left(\tfrac{1}{2}(5x+7)^{-\frac{1}{2}}\right)(5).$$

3. Use the *definition* of the derivative to find the derivative of $f(x) = 3x^2 + 2x - 1$.

$$\lim_{h\to 0} \frac{3(x+h)^2 + 2(x+h) - 1 - (3x^2+2x-1)}{h}$$

$$= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 + 2x + 2h - 1 - 3x^2 - 2x + 1}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2 + 2h}{h}$$

$$= \lim_{h\to 0} 6x + 3h + 2$$

$$= 6x + 2.$$

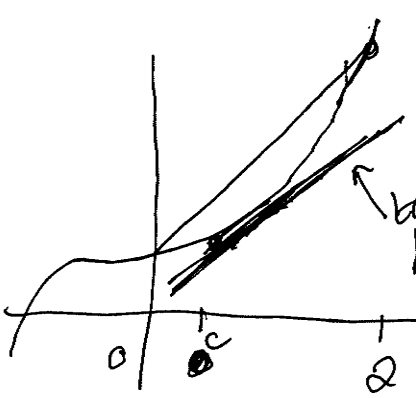
$$\boxed{\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)}$$

4. What does the Mean Value Theorem tell us about $y = x^3 + 1$ on the interval $[0, 2]$?

There exists $c \in [0, 2]$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{9-1}{2} = 4.$$

$f'(c) = 4.$



tangent line at c has same slope as average slope on interval.

$(f'(x) = 3x^2$

$3x^2 = 4$?

$x^2 = \frac{4}{3}$

$x = \pm \frac{2}{\sqrt{3}}$

$c = \frac{2}{\sqrt{3}}$ works $)$

f continuous on $[a, b]$
f differentiable on (a, b) $\Rightarrow$ $\exists\, c \in (a, b)$ s.t. $f'(c) = \frac{f(b) - f(a)}{b - a}.$

6. Find the equation of the tangent line to the curve $x^2 + y^2 = \overset{\frac{5}{2}}{\cancel{2}}xy$ at the point $(1, \overset{2}{\cancel{1}})$.

Implicit Differentiation

$$2x + 2y\frac{dy}{dx} = \underbrace{\tfrac{5}{2}x\frac{dy}{dx} + \tfrac{5}{2}y}_{\text{Product Rule.}}$$

at $(1,1)$:

$$\cancel{2 + 2\frac{dy}{dx} = 2\frac{dy}{dx}} \qquad \frac{dy}{dx} = \frac{\frac{5}{2}y - 2x}{2y - \frac{5}{2}x}$$

at $(1,2)$:

$$\frac{dy}{dx} = \frac{5-2}{4-\frac{5}{2}} = \frac{3}{\frac{3}{2}} = 2$$

$$y - 2 = 2(x-1).$$

7. Use linear approximation to estimate $\sqrt{26}$.

$$f(x) = \sqrt{x}$$

$$f(25) = 5$$

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}} \qquad f'(25) = \frac{1}{10}.$$

$$\sqrt{26} \approx 5 + \frac{1}{10}(26-25) = 5.1$$

(Use Tangent line at $(25,5)$).

8. Find the maximum and minimum values attained by $f(x) = (x^2 - 1)^{2/3}$ on the interval $[-1, 3]$.

$$f'(x) = \frac{2}{3}(x^2-1)^{-\frac{1}{3}}(2x).$$

$f'(x) = 0$? or Undefined?

$x = 0$ $\qquad\qquad$ $x = \pm 1$.

Check $x = 0, \pm 1, \underline{\underline{-1, 3}}$

Endpoints of interval

$f(0) = 1$

$f(1) = 0$

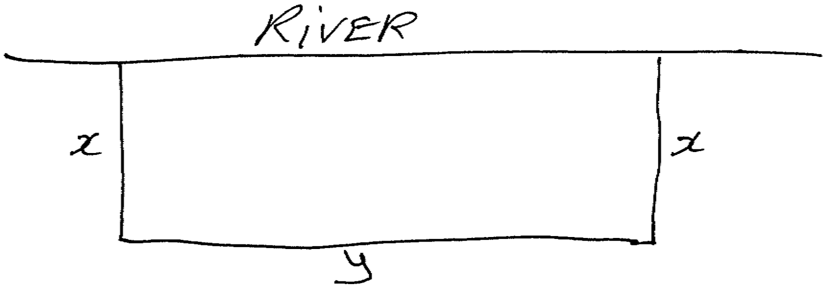
$f(-1) = 0$

$f(3) = (9-1)^{\frac{2}{3}} = (8)^{\frac{2}{3}} = 4.$

Max at $x = 3$, $f(3) = 4$.

Min at ± 1, $f(\pm 1) = 0$.

10. A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. She needs no fence along the river. What are the dimensions of the field that has the largest area?



$2x + y = 2400$, $y = 2400 - 2x$

To maximize Area $= xy$

$A(x) = x(2400 - 2x)$

$= 2400x - 2x^2$

$A'(x) = 2400 - 4x$

$A'(x) = 0 \Rightarrow x = 600.$

$x = 600 \Rightarrow y = 2400 - 2(600) = 1200.$

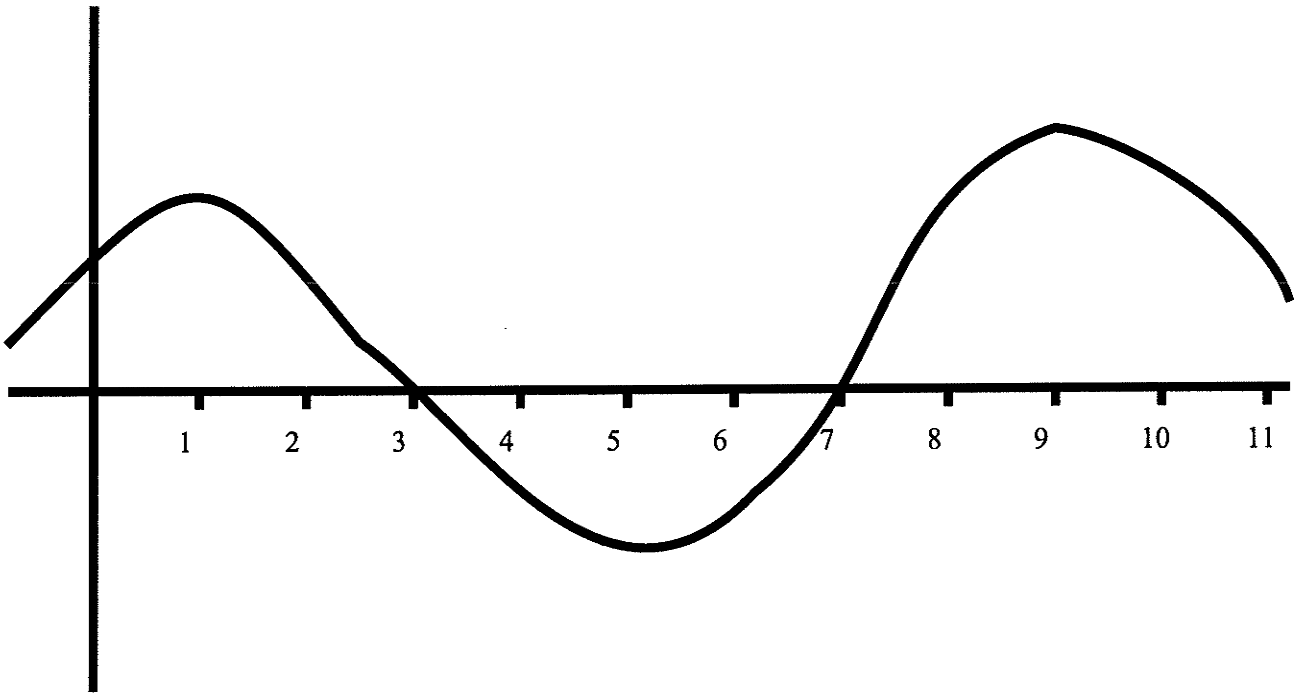
Area $= 600 \times 1200$ ft^2.

Endpoints? Require $\begin{matrix} x \geq 0 \\ y \geq 0 \end{matrix}$, $x \in [0, 1200]$

$A(0) = 0$, $A(1200) = 0.$

Max at $x = 600$, $y = 1200$.

9. The following graph is of the function $y = f(x)$. Define $F(x) = \int_0^x f(x)\,dx$.



(a) Where is the function $F(x)$ increasing? Decreasing?

$(-1, 3) \cup (7, 11)$ Increasing $(F'(x) > 0)$

$(3, 7)$ Decreasing $(F'(x) < 0)$

(b) Where is the function $F(x)$ concave up? Concave down?

Concave up : $F''(x) > 0$, $F'(x) > 0$, $f(x)$ Increasing.

$(-1, 1)$, $(-5, 9)$ Concave up

Concave Down : $(1, 5) \cup (9, 11)$.

(c) Where are the inflection points of $F(x)$?

1, 5

Changes from Concave up to Concave Down.

(d) Classify the critical points of $F(x)$?

3 : Local Max (1st Derivative Test)

7 : Local Min (" " ")

11. Find and classify the critical points of the function

$$y = \frac{x^2}{x+3}$$

$$\frac{dy}{dx} = \frac{(x+3)2x - x^2(1)}{(x+3)^2} = \frac{2x^2+6x-x^2}{(x+3)^2}$$

$$= \frac{x^2+6x}{(x+3)^2}$$

$\frac{dy}{dx} = 0$ or undefined?

$x^2+6x=0 \Rightarrow x=0,-6.$

Undefined at $x=-3$.

At $x=-3$: Vertical asymptote.

$$\frac{d^2y}{dx^2} = \frac{(x+3)^2(2x+6)-(x^2+6x)(2(x+3))}{(x+3)^4}$$

Second Derivative Test.

At $x=0$,

$$\frac{d^2y}{dx^2} = \frac{9(6)-0}{81} = \frac{54}{81} > 0$$

So at $x=0$, local minimum

At $x=-6$,

$$\frac{d^2y}{dx^2} = \frac{9(-6)-0}{81} = -\frac{54}{81} < 0$$

So at $x=-6$, local maximum

12. Evaluate the following integrals.

(a) $\int \frac{1}{x(\ln x)^2}\, dx$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}\, dx$

$= \int \frac{1}{u^2}\, du$

$= -u^{-1} + C$

$= -\frac{1}{\ln x} + C.$

(b) $\int x^3 + \frac{1}{x} + \frac{1}{\sqrt{1-x^2}} + \cos 4x\; dx$

$\frac{1}{4}x^4 + \ln|x| + \arcsin x + \frac{1}{4}\cos 4x + C$

(c) $\int_0^1 5xe^{x^2}\, dx$

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\, dx$

Limits?

$x = 0 \Rightarrow u = 0$

$x = 1 \Rightarrow u = 1.$

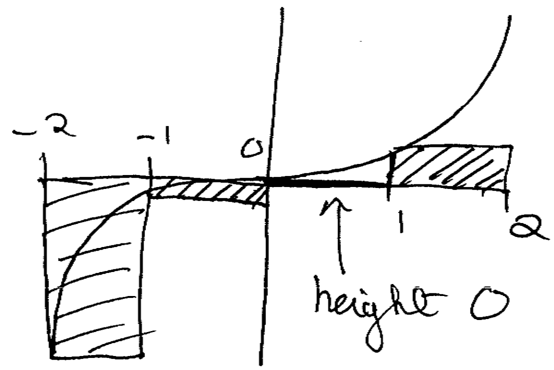
$\frac{1}{2}\, du = x\, dx$

$\frac{1}{2}\int_0^1 5e^u\, du$

$= \frac{5}{2}e^u\Big|_0^1 = \frac{5}{2}(e^1 - 1).$

13. Use a left endpoint Riemann sum with $n = 4$ to estimate

$$\int_{-2}^{2} x^3 dx.$$



Left Endpoints

$-2, -1, 0, 1$

$1\left((-2)^3 + (-1)^3 + 0^3 + 1^3\right)$

$\uparrow$

Δx

$= 1(-8 - 1 + 0 + 1) = -8.$

14. Suppose that a cylindrical block of ice is melting at the rate of 1 cm^3/s, in such a way that the radius is always twice the height. At what rate is the radius changing when the height is 10 cm?

Volume $= 2\pi r h.$ $\quad 2h = r, \quad h = \frac{r}{2}$

$V(t) = 2\pi r\left(\frac{r}{2}\right) = \pi r^2$

$V'(t) = 2\pi r \frac{dr}{dt} \leftarrow$ Solve for $\frac{dr}{dt}$ when $h = 10$

When $h = 10, \quad r = 20$

$1 = 2\pi(20)\frac{dr}{dt}$

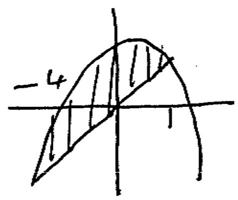
$\frac{1}{40\pi}$ cm/s $= \frac{dr}{dt}.$

15. Find the area bounded between the two curves $y = 4 - x^2$ and $y = 3x$

Intersect when $4 - x^2 = 3x$

$$x^2 + 3x - 4 = 0$$

$$(x+4)(x-1) = 0 \quad x = -4 \text{ or } x = 1.$$

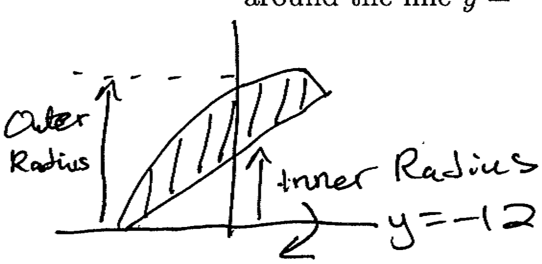


$$\int_{-4}^{1} 4 - x^2 - 3x \, dx$$

$$= 4x - \tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 \Big|_{-4}^{1}$$

$$= 4 - \tfrac{1}{3} - \tfrac{3}{2} - \left(-16 + \tfrac{64}{3} - \tfrac{3}{2}(16)\right)$$

16. Find the volume of the solid obtained by revolving the region from the last problem around the line $y = -12$.



We get a washer for every x between -4 and 1.

Outer Radius $= 4 - x^2 - (-12) = 16 - x^2$

Inner Radius $= 3x - (-12) = 3x + 12$

$$\int_{-4}^{1} \pi (16 - x^2)^2 - \pi (3x + 12)^2 \, dx$$

$$= \pi \int_{-4}^{1} 256 - 32x^2 + x^4 + 9x^2 + 72x + 144 \, dx$$

$$= \pi \left(256x - \tfrac{32}{3}x^3 + \tfrac{x^5}{5} + 3x^3 + 36x^2 + 144x\right) \Big|_{-4}^{1}$$