A Hausdorff-Young Inequality for Locally Compact Quantum Groups

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The Hausdorff-Young Inequality for Locally Compact Abelian Groups

Let $G$ be a locally compact abelian group.
Let $\hat{G}$ be its dual group, i.e.,

$$\hat{G} = \{ \xi : G \to \mathbb{T} \mid \text{continuous homomorphisms} \},$$

- group operation is pointwise multiplication of functions
- $\xi_i \to \xi$ in $\hat{G}$ if $\xi_i$ converges uniformly to $\xi$ on all compact subsets of $G$. 
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\]

- \( \hat{G} = G \) (Pontryagin Duality Theorem)
- Being locally compact groups, \( G \) and \( \hat{G} \) have Haar measures \( \mu \) and \( \hat{\mu} \).
The Hausdorff-Young Inequality for Locally Compact Abelian Groups

Let $G$ be a locally compact abelian group. Let $\hat{G}$ be its dual group, i.e.,

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- $\hat{\hat{G}} = G$ (Pontryagin Duality Theorem)
- Being locally compact groups, $G$ and $\hat{G}$ have Haar measures $\mu$ and $\hat{\mu}$.
The Fourier Transform $\mathcal{F} : L_1(G) \to L_\infty(\hat{G})$ is defined by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_G f(s) \overline{\langle s, \xi \rangle} \, d\mu(s).$$

Clearly,

$$\|\hat{f}\|_\infty \leq \|f\|_1.$$

Also, for $f \in L_1(G) \cap L_2(G),$

$$\|\hat{f}\|_2 = \|f\|_2$$

and $\mathcal{F}$ extends to a unitary $L_2(G) \to L_2(\hat{G})$.
By Riesz-Thorin / complex interpolation method, we have that for $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$
\mathcal{F} : L_p(G) \to L_q(\hat{G}) \\
\|\hat{f}\|_q \leq \|f\|_p.
$$

- Dual Group
- $L_p$-spaces
- Fourier Transform
- Complex Interpolation
Locally compact abelian group $G$

For $f \in L_1(G)$, the operator $\hat{f} \in L_\infty(\hat{G})$ acts by multiplication on $L_2(\hat{G})$:

$$\mathcal{F}(f)\hat{g} = \hat{f}\hat{g} = f \ast g.$$

Thus $\mathcal{F}$ is unitarily equivalent to $\lambda(f)$, left convolution by $f$.

If $G$ is not abelian, we define for $f \in L_1(G), g \in L_2(G)$,

$$\mathcal{F}(f)g = \lambda(f)g = f \ast g.$$

(Kunze, 1958)
The dual object is now the group von Neumann algebra

\[ L(G) = \{ \lambda(f) \mid f \in L_1(G) \}'' \subset B(L_2(G)), \]

which we still denote by \( L_\infty(\hat{G}) \).

- \( L_p(\hat{G}) \) is now a non-commutative \( L_p \)-space.
$M$, von Neumann algebra, with a normal semifinite faithful weight $\varphi$.

- $\mathcal{N}_\varphi = \{ x \in M \mid \varphi(x^*x) < \infty \}$
- $\mathcal{M}_\varphi = \text{span}\{ y^*x \mid x, y \in \mathcal{N}_\varphi \}$
- $\Lambda : \mathcal{N}_\varphi \to H_\varphi$, $(\Lambda(x) \mid \Lambda(y)) = \varphi(y^*x)$
- $M$ acting on $H_\varphi$:
  \[ x\Lambda(y) = \Lambda(xy) \]
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Von Neumann Algebra Preliminaries

\( M \), von Neumann algebra, with a normal semifinite faithful weight \( \varphi \).

- S closure of the map \( \Lambda(x) \mapsto \Lambda(x^*) \), \( x \in \mathcal{M}_\varphi \)
- Polar Decomposition \( S = J\Delta^{1/2} \),
  \( J^2 = 1 \), \( \Delta > 0 \) is self-adjoint and invertible
- Modular automorphism group, \( \sigma^\varphi_t(x) = \Delta^{it}x\Delta^{-it} \), \( t \in \mathbb{R} \)
- \( x \in M \) analytic if \( t \mapsto \sigma^\varphi_t(x) \) extends to an entire function \( \mathbb{C} \to M \).
- Tomita algebra \( \mathcal{A}_0 \subset M \), \( \mathcal{A}'' = M \)
  \( \sigma_z(x) \in \mathcal{M}_\varphi \cap \mathcal{M}_\varphi^* \), \( \forall z \in \mathbb{C} \)
Connes / Hilsum construction
Isometrically isomorphic to Haagerup’s construction
$M$, a von Neumann algebra with a distinguished normal finite
semifinite weight $\varphi$.

Fix a nfs weight $\varphi'$ on $M'$ (e.g., $\varphi'(x) = \varphi(JxJ)$), $H = H_{\varphi'}$,

$$D(H, \varphi') = \left\{ \xi : \exists c \geq 0 \text{ s.t. } \|y\xi\|^2 \leq c\varphi'(y^*y), \forall y \in M' \right\}$$

For each $\xi \in D(H, \varphi')$,

$$\Lambda_{\varphi'}(y) \mapsto y\xi$$

extends to a bounded map $R_{\varphi'}(\xi) \in M$. 

\[\text{8}\]
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Normal linear functional $\psi \in M_*$, spatial derivative $\frac{d\psi}{d\varphi'}$ is the unique, self-adjoint operator on $H$ such that for $\xi \in D(H, \varphi')$, 

$$\left\| \left( \frac{d\psi}{d\varphi'} \right)^{1/2} \xi \right\|^2 = \varphi(R_{\varphi'}(\xi)R_{\varphi'}(\xi)^*).$$
\[ L_1(M, \varphi') = \{ \text{closed operators } T = u|T| \text{ such that} \]
\[ u \in M, \exists \psi \in M^+_* \text{ such that } |T| = \frac{d\psi}{d\varphi'} \}
\[ \|T\|_1 = \psi(1) = \|\psi\|_1 \]

\( L_1(M, \varphi') \) is isometrically isomorphic to \( M_* \).

\[ L_p(M, \varphi') = \{ \text{closed operators } T = u|T| \text{ such that} \]
\[ u \in M, \exists \psi \in M^+_* \text{ such that } |T|^p = \frac{d\psi}{d\varphi'} \}
\[ \|T\|_p = \| |T|^p \|_1^{1/p} = \psi(1)^{1/p} \]
If $\varphi$ is a fixed nfs weight on $M$, set

$$d = \frac{d\varphi}{d\varphi'}.$$ 

$$\left\{ d^{1/2p} y^* x d^{1/2p} \mid x, y \in \mathcal{M}_\varphi \right\}_{\|\cdot\|^p} = L_p(M, \varphi'), 1 \leq p < \infty$$

$$\left\{ x d^{1/p} \mid x \in \mathcal{M}_\varphi \right\}_{\|\cdot\|^p} = L_p(M, \varphi'), 2 \leq p < \infty$$
For $x \in \mathcal{A}_0$, $0 \leq \alpha \leq \frac{1}{2}$,

$$xd^\alpha = d^\alpha \sigma^\varphi_{i\alpha}(x).$$

$$d \longleftrightarrow \varphi$$

$$xd \longleftrightarrow \varphi(\cdot x)$$

$$d^{1/2} \sigma^\varphi_{i/2}(x)d^{1/2} \longleftrightarrow \psi$$ such that

$$\frac{d^\psi}{d^\varphi'} = d^{1/2} \sigma^\varphi_{i/2}(x)d^{1/2},$$

where $\psi = \varphi(\cdot x) \in L_1(\mathbb{G})$ satisfies

$$\langle \varphi(\cdot x), y \rangle = \varphi(yx),$$

for $y \in \mathcal{M}_\varphi$, $x \in \mathcal{A}_0^2$. 
$G$, a (not necessarily abelian) locally compact group $G$, with modular function $\Delta$.

What are $L_p(\hat{G}) = L_p(L(G))$?

We view $\Delta$ as an unbounded operator on $L_2(G)$:

$$(\Delta \xi)(t) = \Delta(t)\xi(t)$$

$\text{Domain}(\Delta) = \{ \xi \in L_2(G) \text{ such that } \Delta(t)\xi(t) \in L_2(G) \}$$
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Hausdorff-Young for locally compact groups

$G$, a (not necessarily abelian) locally compact group $G$, with modular function $\Delta$.

$$\hat{d} = \frac{d\hat{\varphi}}{d\hat{\varphi}'} = \Delta,$$

(where $\hat{\varphi}' = \hat{\varphi}(\hat{J} \cdot \hat{J})$, $\hat{\varphi}$ the Plancherel weight on $L(G)$

$$\hat{\varphi}(\lambda(f)^* \lambda(g)) = (g \mid f),$$

for $f, g \in L_1(G) \cap L_2(G)$.)
Theorem (Terp, 1980)

For $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L_p(G)$,

$$\mathcal{F}_p(f)\xi := f \ast \Delta^{1/q}\xi,$$

with domain $\{\xi \in L_2(G) \text{ such that } f \ast \Delta^{1/q}\xi \in L_2(G)\}$.

$$\mathcal{F}_p(f) \in L_q(\hat{G})$$

We could also write this as

$$\mathcal{F}_p(f) = \lambda(f)\hat{a}^{1/q}$$
Locally Compact Quantum Groups

A locally compact quantum group $G = (M, \Gamma, \varphi, \psi)$ consists of

- a von Neumann algebra $M$
- a normal, unital, $*$-homomorphism $\Gamma : M \to M \otimes M$ with
  \[
  (\Gamma \otimes \iota) \circ \Gamma = (\iota \otimes \Gamma) \circ \Gamma
  \]
- a nsf weight $\varphi$ on $M$ such that for $\omega \in M_+^*, x \in M_\varphi^+$,
  \[
  \varphi((\omega \otimes \iota) \Gamma(x)) = \varphi(x) \omega(1)
  \]
- a nsf weight $\psi$ on $M$ such that for $\omega \in M_+^*, x \in M_\psi^+$,
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(Kustermans and Vaes)
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\[\Gamma : L_\infty(G) \to L_\infty(G \times G)\]
\[\Gamma(f)(s, t) = f(st)\]
\[f((st)u) = f(s(tu))\]

- a nsf weight $\varphi$ on $M$ such that for $\omega \in M_\varphi^+, x \in M_\varphi^+$,
  \[\varphi((\omega \otimes \iota)\Gamma(x)) = \varphi(x)\omega(1)\]

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  $$\psi((\iota \otimes \omega)\Gamma(x)) = \psi(x)\omega(1)$$
  (Kustermans and Vaes)
We will write $L_\infty(G)$ for $M$ and $L_1(G)$ for $M^*$

Multiplicative unitary $W$ on $H \otimes H$

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Gamma(y)(x \otimes 1))$$

for all $x, y \in \mathcal{M}_\varphi$.

$$\Gamma(x) = W(1 \otimes x)W^*$$

Group case: $W$ unitary on $L_2(G \times G)$

$$W(f \otimes g)(s, t) = f(s)g(s^{-1}t)$$
Fourier Representation of $L_1(G)$

Multiplication on $M_* = L_1(G)$:

$$\langle \omega_1 * \omega_2, x \rangle = \langle \omega_1 \otimes \omega_2, \Gamma(x) \rangle$$

Fourier Representation of $L_1(G)$:

$$\lambda(\omega) = (\omega \otimes \iota)(W), \quad \omega \in L_1(G)$$

$$\lambda(\omega_1 * \omega_2) = \lambda(\omega_1) \lambda(\omega_2)$$
Dual Locally Compact Quantum Group

Dual LCQG is
\[ \hat{G} = (L_\infty(\hat{G}), \hat{\Gamma}, \hat{\phi}, \hat{\psi}), \]
where \( L_\infty(\hat{G}) = \lambda(L_1(G))'' \).

Multiplicative unitary \( \hat{W} = \Sigma W^* \Sigma \), where \( \Sigma(x \otimes y) = y \otimes x \),
\[ \hat{\Gamma}(\hat{x}) = \Sigma W^*(\hat{x} \otimes 1) W \Sigma. \]

Pontryagin Duality Theorem:
\[ \hat{\hat{G}} = G \]
Haar Weight on Dual LCQG

\[ \mathcal{I} = \{ \omega \in L_1(\mathbb{G}) \mid \exists c \in \mathbb{R}^+ : |\omega(x^*)| \leq c \|\Lambda(x)\| \text{ for all } x \in \mathcal{M}_\varphi \}. \]

By the Riesz Representation Theorem, there then exists \( \xi(\omega) \in H \) such that

\[ \omega(x^*) = (\xi(\omega) \mid \Lambda(x)), \quad x \in \mathcal{M}_\varphi. \]

\( \mathcal{I} \) is a left ideal in \( L_1(\mathbb{G}) \) with \( \lambda(\omega_1)\xi(\omega_2) = \xi(\omega_1 \ast \omega_2) \).
\( \lambda(\mathcal{I}) \) is a \( \sigma \)-strong*-norm core for the unique \( \sigma \)-strong*-norm closed linear map \( \hat{\Lambda} \) such that

\[
\hat{\Lambda}(\lambda(\omega)) = \xi(\omega).
\]

The dual weight \( \hat{\varphi} \) is the unique nsf weight on \( \hat{M} \) with the triple \( (H, \iota, \hat{\Lambda}) \) as its GNS construction. Here \( \iota \) is the action

\[
\lambda(\omega_1)\xi(\omega_2) = \xi(\omega_1 \ast \omega_2).
\]

Note that \( \omega \in \mathcal{I} \) implies that \( \lambda(\omega) \in \mathcal{N}_{\hat{\varphi}} \).
For $p = 1$, the definition is clear: for $\omega \in L_1(\mathbb{G})$,

$$F_1(\omega) = \lambda(\omega) \in L_\infty(\hat{\mathbb{G}}),$$

and

$$\|F_1(\omega)\|_\infty = \|(\omega \otimes \iota)(W)\|_\infty \leq \|\omega\|_1.$$  

Abusing notation, we can write (e.g., for $x \in \mathfrak{A}_0^2 \subset L_\infty(\mathbb{G})$):

$$F_1(\varphi(\cdot x)) = F_1(xd') = \lambda(\varphi(\cdot x))$$
\( p = 2: \) for \( x \in A_0^2 \subset L_\infty(G), \)

\[
\begin{align*}
H_\varphi & \sim L_2(G) \sim L_2(\hat{G}) \sim H_\hat{\varphi} \\
\Lambda(x) & \mapsto xd^{1/2} \mapsto \lambda(\varphi(\cdot x))d^{1/2} \mapsto \hat{\Lambda}(\lambda(\varphi(\cdot x)))
\end{align*}
\]

\[
\mathcal{F}_2(xd^{1/2}) = \lambda(\varphi(\cdot x))d^{1/2}
\]

and

\[
\|\mathcal{F}_2(xd^{1/2})\|_2 = \|\lambda(\varphi(\cdot x))d^{1/2}\|_2
\]
For $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $x \in \mathbb{A}_0$,

\[
\mathcal{F}_p : L_p(G) \rightarrow L_q(\hat{G})
\]

\[
d^{1/2p} \sigma_{i/2p}(x) d^{1/2p} \mapsto \lambda(\varphi(\cdot x)) \hat{d}^{1/q}
\]

\[
xd^{1/p} \mapsto \lambda(\varphi(\cdot x)) \hat{d}^{1/q}
\]
With suitable embeddings into a Banach space, the complex interpolation method yields:

$$L_p(M) = (M, M_*)_{1/p}$$

If $x \in M \cap M_*$, what is the corresponding element of $L_p(M)$?

- **Kosaki**: For a state $\varphi$, $M \ni x \mapsto d^{\eta/2p} x d^{(1-\eta)/2p}$, $\eta \in [0, 1]$
- **Terp**: For a weight $\varphi$, $\mathcal{M}_\varphi \ni x \mapsto d^{1/2p} x d^{1/2p}$
- **Izumi**: Generalizes both Terp and Kosaki.
For each $\alpha \in \mathbb{C}$, we define

$$L_\alpha = \left\{ x \in M \mid \text{there exists a unique functional } \varphi_\alpha^x \in M_* \text{ s.t.} \right. $$

$$\varphi_\alpha^x(y^* z) = (xJ\Delta^{\alpha} \Lambda(y) \mid J\Delta^{-\alpha} \Lambda(z))$$

for all $y, z \in \mathcal{A}_0$.

This is a Banach space when considered with the norm

$$\|x\|_{L_\alpha} = \max\{\|x\|_\infty, \|\varphi_\alpha^x\|_1\},$$

where $\| \cdot \|_\infty$ and $\| \cdot \|_1$ are the norms on $M$ and $M_*$, respectively.
For any $\alpha \in \mathbb{C}$, we have

$$A^2_0 \subset L_\alpha$$

and for $y, z \in A_0$, we have

$$\varphi^\alpha_{y^*z} = \omega J\Delta^{-\bar{\alpha}} \Lambda(y), J\Delta^\alpha \Lambda(z)$$

where $\omega_{\xi, \eta}$ is the functional $(\cdot \xi | \eta)$. 
The maps

\[ i_\alpha : L_\alpha \rightarrow M, \ x \mapsto x \text{ and } j_\alpha : L_\alpha \rightarrow M^*_\alpha, \ x \mapsto \varphi^\alpha_x, \]

are norm-decreasing and injective. The set \( i_\alpha(\mathbb{A}_0^2) \) is \( \sigma \)-weakly dense in \( M \) and the set \( j_\alpha(\mathbb{A}_0^2) \) is norm dense in \( M^*_\alpha \).
Izumi’s interpolation result when $\alpha = -1/2$
Hausdorff-Young Inequality for LCQG

- Holds for $F_1 : L_1(G) \to L_\infty(\hat{G})$.
- Holds for $F_2 : L_2(G) \to L_2(\hat{G})$.
- Do $F_1$ and $F_2$ agree on $L_1(G) \cap L_2(G)$?
- Check $F_1$ and $F_2$ well-defined and agree on images of $x \in \mathcal{A}^2_0$ in $L^*_{1/2}$.
- $L_1(G) \cap L_2(G) \subset L^*_{1/2}$
- $L_2(\hat{G}) \cap L_\infty(\hat{G}) \subset \hat{L}^*_{1/2}$
- Approximation arguments
- Interpolation
Theorem (C)

Locally compact quantum group $G$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\mathcal{F}_p : L_p(G) \rightarrow L_q(\hat{G})$$

$$xd^{1/p} \mapsto \lambda(\varphi(\cdot x))\hat{d}^{1/q}$$

$$\|\mathcal{F}_p(xd^{1/p})\|_q \leq \|xd^{1/p}\|_p$$
Thanks!!

Thanks for your attention!

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