RAMSEY FUNCTIONS FOR GENERALIZED PROGRESSIONS

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Abstract
Given positive integers $m$ and $k$, a $k$-term semi-progression of scope $m$ is a sequence $x_1, x_2, \ldots, x_k$ such that $x_{j+1} - x_j \in \{d, 2d, \ldots, md\}$, $1 \leq j \leq k - 1$, for some positive integer $d$. Thus an arithmetic progression is a semi-progression of scope $1$. Let $S_m(k)$ denote the least integer for which every 2-coloring of $\{1, 2, \ldots, S_m(k)\}$ yields a monochromatic $k$-term semi-progression of scope $m$. We obtain an exponential lower bound on $S_m(k)$ for all $m = O(1)$. Our approach also yields a marginal improvement on the best known lower bound for the analogous Ramsey function for quasi-progressions, which are sequences whose successive differences lie in a small interval.

1. Introduction
In 1927, B.L. van der Waerden [6] proved that given positive integers $r$ and $k$, there exists an integer $W(r, k)$ such that any $r$-coloring of $\{1, 2, \ldots, W(r, k)\}$ yields a monochromatic $k$-term arithmetic progression. Even after nearly 90 years, the gap between the lower and upper bounds on $W(r, k)$ remains enormous, with the best known lower bound of the order of $r^k$, whereas the best known upper bound is a five-times iterated tower of exponents (see [1]). Analogues of the Van der Waerden threshold $W(r, k)$ have been studied for many variants of arithmetic progressions, including semi-progressions and quasi-progressions (see [4]).

Let $m$ and $k$ be positive integers. A $k$-term semi-progression of scope $m$ is a sequence $x_1, x_2, \ldots, x_k$ such that for some positive integer $d$, $x_{j+1} - x_j \in \{d, 2d, \ldots, md\}$. The integer $d$ is called the low-difference of the semi-progression. We define $S_m(k)$ as the least integer for which any 2-coloring of $\{1, 2, \ldots, S_m(k)\}$ yields a monochromatic $k$-term semi-progression of scope $m$. Note that $S_m(k) \leq W(k)$ with equality if $m = 1$. 
2. An Exponential Lower Bound for $S_m(k)$

Landman [3] showed that $S_m(k) \geq (2k^2/m)(1 + o(1))$. We improve this to an exponential lower bound for all $m = O(1)$.

**Theorem 1** Let $k \geq 3$, $m = O(1)$ and $\alpha = \sqrt{2^m/(2^m - 1)}$. Then $S_m(k) > \alpha^k$.

**Proof.** Let $f(N, k, m)$ denote the number of 2-colorings of $[1, N]$ with a monochromatic $k$-term semi-progression of scope $m$. (In the remainder of the proof, we only consider $k$-term semi-progressions of scope $m$.) Note that $S_m(k)$ is the least integer $N$ such that $f(N, k, m) = 2^N$. We derive an upper bound on $f(N, k, m)$ as follows.

Given a semi-progression $P = (a_1, a_2, \ldots, a_k)$ of low-difference $d$, we define the conjugate vector of $P$ as $(u_1, u_2, \ldots, u_{k-1})$ where $u_i = (a_{i+1} - a_i - d)/d$. Likewise, the frequency vector of $P$ is defined as $v = (v_0, v_1, \ldots, v_{m-1})$ where $v_j$ is the number of times $j$ occurs in the conjugate vector of $P$. Note that $\Sigma_{j=0}^{m-1} v_j = k - 1$. Finally, the weight of a frequency vector $v$, denoted $w(v)$, is defined as $\sum_{j=0}^{m-1} jv_j$.

Given a coloring $\chi$, we define the $(a, d)$-primary semi-progression of $\chi$ as the semi-progression $P$ whose conjugate vector is lexicographically least among the conjugate vectors of all semi-progressions (with first term $a$ and low-difference $d$) that are monochromatic under $\chi$. Let $P = (a_1, a_2, \ldots, a_k)$ be a semi-progression with first term $a_1 = a$ and low-difference $d$. We will give an upper bound for the number of colorings $\chi$ such that $P$ is the $(a, d)$-primary semi-progression of $\chi$.

Since $P$ is monochromatic, all elements of $P$ have the same color under $\chi$. Furthermore, if $(v_0, v_1, \ldots, v_{m-1})$ is the frequency vector of $P$, it follows from the fact that $P$ is the $(a, d)$-primary semi-progression of $\chi$ that $w(v)$ elements in the arithmetic progression $\{a, a + d, \ldots, a + m(k-1)d\}$ must be of the color different from the color of the elements of $P$. For example, let $a = 17, d = 5, m = 3, k = 6$ and $P = \{17, 32, 42, 47, 62, 72\}$ with conjugate vector $(2, 1, 0, 2, 1)$. If the two colors are red and blue, and the elements of $P$ are all red, then 22, 27, 37, 52, 57 and 67 must all be blue. Indeed, if 57 is red, then the semi-progression $P' = \{17, 32, 42, 47, 57, 62\}$ would have a lexicographically lower conjugate vector $(2, 1, 0, 1, 0)$. Thus there are at most $2^{N-1}$ colorings of $[1, N]$ whose $(a, d)$-primary semi-progression is $P$.

Note that there are at most $N^2/(k-1)$ choices for the pair $(a, d)$. We say that two progressions $P_1$ and $P_2$ with the same $a$ and $d$ are equivalent if they have the same frequency vector. Note that for any $a$ and $d$, there are at most

$$M(v) = \frac{(m + v_1 + \cdots + v_{m-1})!}{v_0!v_1!\cdots v_{m-1}!}$$

semi-progressions with frequency vector $(v_0, v_1, \ldots, v_{m-1})$. Adding over all the
of low difference. We now apply the same technique to quasi-progressions. A $k$-term quasi-progression of diameter $d$ and diameter $n$ is a sequence $(a_1, a_2, \ldots, a_k)$ such that $d \leq a_{j+1} - a_j \leq d + n, 1 \leq j \leq k - 1$. Let $Q_n(r, k)$ denote the least positive integer such that any $r$-coloring of $\{1, 2, \ldots, Q_n(r, k)\}$ yields a monochromatic $k$-term quasi-progression of diameter $n$. It is known (see [5]) that $Q_1(2, k) > \beta^k$ where $\beta = 1.08226$... Indeed, $\beta^2$ can be expressed in terms of two algebraic numbers of degrees 2 and 3, respectively, and is the smallest positive real root of the equation $y^6 + 8y^5 - 112y^4 - 128y^3 + 1792y^2 + 1024y - 4096 = 0$. It is also known that $Q_n(2, k) = O(k^2)$ for $n > k/2$ (see [2]). We apply the techniques of the previous section to obtain lower bounds on $Q_n(r, k)$. Let $g(r, N, k, n)$ denote the number of $r$-colorings of $[1, N]$ with a monochromatic $k$-term semi-progression of diameter $n$. Note that $Q_n(r, k)$ is the least positive integer $N$ such that $g(r, N, k, n) = r^N$. We first discuss the simplest non-trivial case, namely $r = 2$ and $n = 1$.

**Theorem 2** Let $k \geq 3$. Then $Q_1(2, k) > \beta_{2,1}^k$ where $\beta_{2,1} = \sqrt[4]{4-2\sqrt{2}} = 1.08239$...

**Proof.** We define the conjugate vector of a quasi-progression $Q = \{a_1, a_2, \ldots, a_k\}$ of low-difference $d$ as $(u_1, u_2, \ldots, u_{k-1})$ where $u_i = a_{i+1} - a_i - d$. Given a coloring $\chi$, we define the $(a, d)$-primary quasi-progression of $\chi$ as the quasi-progression $Q$ whose conjugate vector is lexicographically least among the conjugate vectors of all quasi-progressions (with first term $a$ and low-difference $d$) that are monochromatic under $\chi$. Let $Q = \{a_1, a_2, \ldots, a_k\}$ be a quasi-progression with first term $a_1 = a$ and low-difference $d$. We give an upper bound for the number of colorings $\chi$ such that $Q$ is the $(a, d)$-primary quasi-progression of $\chi$.

Since $Q$ is monochromatic, all elements of $Q$ have the same color under $\chi$, say red. Let $(u_1, u_2, \ldots, u_{k-1})$ be the conjugate vector of $Q$. Observe that if $u_j = 1$
and \( u_{j+1} = 0 \) for some \( j \), so that \( a_j, a_j + d + 1 \) and \( a_j + 2d + 1 \) are elements of \( Q \), and therefore red, it follows that the color of \( a_j + d \) is different from red (say blue), as \((P \cup \{a_j + d\}) \setminus \{a_j + d + 1\}\) has a lexicographically lower conjugate vector. We define the weight of \( Q \), denoted \( w(Q) \), as the sum of the last element of the conjugate vector of \( Q \), and the number of occurrences of the string “10” in the conjugate vector of \( Q \). Note that in view of the above observation, the color of \( w(Q) \) integers in the set \( \{a, a + d, a + d + 1, \ldots, a + (k-1)d, \ldots, a + (k-1)(d+1)\} \) can be inferred to be blue.

We now derive an upper bound on \( g(2, N, k, 1) \). There are \( N^2/(k-1) \) choices for the pair \((a, d)\). Of the \( 2^{k-1} \) possible conjugate vectors for a quasi-progression with first term \( a \) and common difference \( d \), let \( w_\ell \) be the number of conjugate vectors of weight \( \ell \). Let

\[
S_t = \sum_{\ell=0}^{\lceil t/2 \rceil} w_\ell 2^{-\ell}
\]

denote the weighted sum of all such vectors of length \( t \). Clearly, \( S_t = S_{t,0} + S_{t,1} \) where \( S_{t,0} \) and \( S_{t,1} \) denote the weighted sum of conjugate vectors that begin with 0 and 1 respectively, with \( S_{1,0} = 1 \) and \( S_{1,1} = 1/2 \). It is easy to see that

\[
A[S_{t-1,0} S_{t-1,1}]^T = [S_{t,0} S_{t,1}]^T
\]

where

\[
A = \begin{bmatrix}
1 & 1 \\
1/2 & 1
\end{bmatrix}
\]

Since \( \lambda_{\text{max}}(A) = 1 + \frac{1}{\sqrt{2}} \), we get

\[
g(2, N, k, 1) < \frac{N^2 2^{N-k+1} \left[ \left( 1 + \frac{1}{\sqrt{2}} \right)^k + \left( 1 - \frac{1}{\sqrt{2}} \right)^k \right]}{2(k-1)}
\]

Thus \( g(2, N, k, 1) < 2^N \) for \( N = \beta_{2,1}^k \) where \( \beta_{2,1} = \sqrt{4 - 2\sqrt{2}} = 1.08239\ldots \) is the smallest positive real root of the equation \( g^4 - 8g^2 + 8 = 0 \). It follows that \( Q_1(2, k) > \beta_{2,1}^k \) yielding a tiny improvement over the lower bound in [5]. \( \square \)

In general, since there are \( r^N \) \( r \)-colorings of \([1, N]\) and at most \( N^2(n+1)^{k-1} \) \( k \)-term quasi-progressions of diameter \( n \), a lower bound of the form \( Q_n(r, k) \geq (\sqrt{r/(n+1)})^k \) follows immediately from the linearity of expectation. However, this bound is only useful when \( n \leq r - 2 \). Generalising the approach outlined earlier, we represent the conjugate vector of \( Q \) as an \( r \)-ary string, and define the weight \( w(Q) \) as the sum of the last element of the conjugate vector of \( Q \), and the number of occurrences of strings of length two of the form “xy”, counted with multiplicity \( m(x, y) = \min(x, n-y) \). (Note that \( m(x, y) \) denotes the number of conjugate vectors that are lexicographically lower than the given vector and correspond to quasi-progressions that differ from \( Q \) in exactly one element.)
As before, let $S_{t,j}$ denote the weighted sum of conjugate vectors of length $t$ beginning with $j$, $0 \leq j \leq n$, with $S_{t,j} = \alpha^j$ for all $j$ where $\alpha = 1 - \frac{1}{r}$. Then $A[S_{t,0} \cdots S_{t,n}]^T = [S_{t+1,0} \cdots S_{t+1,n}]^T$ where

$$A_{r,n} = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
\alpha & \alpha & \cdots & \alpha & 1 \\
\alpha^2 & \alpha^2 & \cdots & \alpha & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha^n & \alpha^{n-1} & \cdots & \alpha & 1
\end{bmatrix}$$

Note that the $(i,j)^{th}$ entry of the matrix $A_{r,n}$ is $\alpha^{\min(i-1,j-1)} = \alpha^{\min(i-1,n+1-j)}$. Then $Q_n(r,k) > \beta^k$ where $\beta = \beta_{r,n} = \sqrt{r/\lambda_{\text{max}}(A_{r,n})}$. Note that for each $r$, there are only finitely many values for which $\beta_{r,n} > 1$. The first few such values are shown in the following table.

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<th>$n$</th>
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<td>1.46410</td>
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References


