1. Find all differentiable functions $f$ defined on positive real numbers satisfying $f(xy) = f(x) + f(y)$ for all $x, y$.

2. Let $0 < x_1 \leq x_2 \leq \ldots \leq x_n < 1$. Prove that at least one of the following inequalities holds:

$$x_1 x_2 \cdots x_n \leq 2^{-n}, \quad (1 - x_1)(1 - x_2) \cdots (1 - x_n) \leq 2^{-n}$$

3. (Putnam ’86 B4) For a positive real number $r$, let $G(r)$ be the minimum value of the expression $|r - \sqrt{m^2 + 2n^2}|$ for all integers $m$ and $n$. Prove or disprove the assertion that $\lim_{r \to \infty} G(r)$ exists and equals 0.

4. (Putnam ’73 B4) Let $f$ be a differentiable function on $(0, 1)$ with $f(0) = 0$ and $0 < f'(t) \leq 1$ for $t \in (0, 1)$. Show that

$$\left( \int_0^1 f(t) \, dt \right)^2 \geq \int_0^1 (f(t))^3 \, dt$$

5. (Putnam ’98 A3) Let $f : \mathbb{R} \to \mathbb{R}$ be a function with continuous third derivative. Prove that there exists a point such that

$$f(a) f'(a) f''(a) f'''(a) \geq 0$$