ON THE LARGEST $k$-PRIMITIVE SUBSET OF $[1,n]$

Sujith Vijay
Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA.
sujith@math.rutgers.edu

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Abstract

We derive bounds on the size of the largest subset of $\{1, 2, \ldots, n\}$ such that no element divides $k$ others, for $k \geq 3$ and sufficiently large $n$.

1. Introduction

Let $S \subseteq \mathbb{N}$ be a finite set of positive integers. We say that $S$ is $k$-primitive if no member of $S$ divides $k$ other elements in $S$.

Let $f_k(n)$ denote the size of the largest $k$-primitive subset of $[1, n]$. It is well-known that $f_1(n) = \left\lceil \frac{n}{2} \right\rceil$. Lebensold [2] showed that, if $n$ is sufficiently large,

$$(0.672...) < \frac{f_2(n)}{n} < (0.673...)$$

In this article, we show that, for $k \geq 3$ and sufficiently large $n$,

$$\frac{k}{k+1} + \frac{1}{8k^4} < \frac{f_k(n)}{n} < 1 - \frac{1}{8k \ln k}$$

Moreover, given $\epsilon > 0$, there exists $k_0(\epsilon)$ such that for $k \geq k_0(\epsilon)$ and $n \geq n_0(k)$,

$$\frac{k}{k+1} + \frac{1 - \epsilon}{k^4} < \frac{f_k(n)}{n} < 1 - \frac{1}{(2e^\gamma + \epsilon)k \ln k}$$

2. The Lower Bound

For $\alpha \in \mathbb{R}$ and $S \subseteq \mathbb{N}$, we shall write $\alpha S$ to denote the set $\{\alpha x : x \in S\}$. We begin by deriving a lower bound on $f_k(n)$. 
Define $S_0 = \{x : (k+1)x > n\}$, with $|S_0| = \frac{nk}{k+1} + O(1)$. Clearly, $S_0$ is $k$-primitive. Let $S_1 = \{x : \frac{n}{k+3} < x < \frac{nk}{(k+1)^2}, k(k+1)|x\}$. Observe that any element in $S_1$ has exactly $k + 1$ other multiples in $[1, n]$. Let $S_2 = (k+1)S_1$, $S_3 = (k+2)S_1$ and $S' = (S_0 \cup S_1) \setminus (S_2 \cup S_3)$. Note that $S'$ is $k$-primitive.

Let $S_4 = (k+1)^{-1}S_3$ and and $S_5 = k^{-1}S_2$. Any element in $S_4 \cup S_5$ has at most $k$ other multiples in $[1, n]$. By construction, at least one of these will not occur in $S'$. Furthermore, no multiple of an element in $S_4$, except possibly itself, occurs in $S_5$ and vice versa. It follows that $S \geq S' \cup S_4 \cup S_5$ is $k$-primitive.

Note that

$$|S_1| = \frac{n(k-1)}{k(k+1)^3(k+3)} + O(1), \text{ for } 1 \leq i \leq 5$$

Furthermore,

$$S_i \cap S_j = \emptyset \text{ for } 1 \leq i < j \leq 5 \text{ except when } i = 4 \text{ and } j = 5.$$ 

Finally,

$$|S_4 \cap S_5| = \frac{n(k^3 - 4k - 1)}{k^2(k+1)^5(k+2)(k+3)} + O(1)$$

Thus we have,

$$|S| = |S_0| + |S_1| - |S_4 \cap S_5| > n \left( \frac{k}{k+1} + \frac{1}{8k^4} \right)$$

Note that for sufficiently large $k$,

$$|S| > n \left( \frac{k}{k+1} + \frac{1-\epsilon}{k^4} \right)$$

3. The Upper Bound

Let $S$ be a $k$-primitive subset of $[1, n]$. For a positive integer $x \leq n/(k+1)$, let $C_x = \{x, 2x, \ldots, (k+1)x\}$ be the chain containing $x$. Observe that $C_x \subseteq [1, n]$ and $|S \cap C_x| \leq k$. Thus if $C_{x_1}, C_{x_2}, \ldots, C_{x_m}$ are pairwise disjoint, $|S| \leq n - m$.

Let $X = \{x : \frac{n}{2(k+1)} < x < \frac{n}{k+1}, x \text{ has no prime factor in } [2, k]\}$. Thus if $r \leq k$ and $x \in X$, we have $(r, x) = 1$. 


We claim that \( \{C_{x_m}\}, x_m \in X \) is a pairwise disjoint collection.

Suppose not. Let \( rx_i = sx_j, x_i \neq x_j, 1 \leq r < s \leq k + 1 \). Since \( r \leq k \) and \( x_j \in X \), we have \( (r, x_j) = 1 \). Thus \( x_j | x_i \), i.e., \( x_i \geq 2x_j > \frac{n}{k+1} \), which is impossible. This proves our claim.

Let \( P_k \) denote the product of the prime numbers not exceeding \( k \). The easy estimate \( P_k < 3^k \), together with an application of the Chinese Remainder Theorem, yields

\[
|X| = \frac{n}{2(k+1)} \prod_{p \leq k} \left( 1 - \frac{1}{p} \right) + O(3^k)
\]

By Mertens’s theorem,

\[
\prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \geq \frac{1}{e^{\gamma+\delta} \ln x} \quad \text{where} \quad |\delta| < \frac{4}{\ln(x+1)} + \frac{1}{2x} + \frac{2}{x \ln x}
\]

Computations for a bounded initial segment (suffices to consider \( x < 12000 \)) establish that

\[
\prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \geq \frac{\ln 3}{3 \ln x} \quad \text{for} \quad x \geq 3
\]

Therefore, we obtain, for \( k \geq 3 \),

\[
|X| > \frac{n}{8k \ln k}
\]

and for sufficiently large \( k \),

\[
|X| > \frac{n}{(2e^\gamma + e)k \ln k}
\]

References

