Longest Cycles in $k$-connected Graphs with Given Independence Number

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- A vertex subset $S \subseteq V(G)$ is said to be independent such that no two vertices are adjacent.
- $\alpha(G)$: the independence number of a graph $G$, the maximum size of an independent set of vertices.
In 1952, Dirac proved that if $G$ is a simple graph with $n$ vertices ($n \geq 3$) and $\delta(G) \geq \frac{n}{2}$, then $c(G) = n$. 
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More generally, in 1960, Ore proved that if a graph $G$ satisfies the property that $d(u) + d(v) \geq n$ whenever $uv \notin E(G)$, then $c(G) = n$. 
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In 1972, Chvátal and Erdős showed that if $\kappa(G) \geq \alpha(G)$ for a graph $G$, then $c(G) = n$. 
In 1952, Dirac also proved that if $G$ is a 2-connected graph with $n$ vertices, then $c(G) \geq \min\{n, 2\delta(G)\}$. 
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More generally, Bondy (1971), and Bermad and Liniar (1976) proved that if a 2-connected graph $G$ satisfies the property that $d(u) + d(v) \geq s$ whenever $uv \notin E(G)$, then $c(G) \geq \min\{n, s\}$. 
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Now, is there a longest cycle version of Chvátal and Erdős theorem?
Longest Cycle Versions for 2-connected Graphs
Conjecture (Fouquest-Jolivet 1976)

If $\kappa(G) \leq \alpha(G)$, then

$$c(G) \geq \frac{k(n + \alpha - k)}{\alpha}$$

, where $k = \kappa(G)$, and $\alpha = \alpha(G)$. 
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In 2009, Manousakis proved it for the case $k = 3$. 
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\[
\begin{align*}
n &= k + \alpha m, \quad \alpha(G) &= \alpha, \quad \kappa(G) &= k, \quad c(G) &= k(1 + m) = \frac{k(n + \alpha - k)}{\alpha}.
\end{align*}
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When Fournier and Manousakis proved for the cases $k = 2$ and $k = 3$, respectively, both of them used the special cases ($k = 2$ and $k = 3$) of the following conjecture.
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**Conjecture (Chen-Chen-Liu)**

If $C_1$ and $C_2$ are distinct cycles in a $k$-connected graph $G$, then there are distinct cycles $C'_1$ and $C'_2$ in $G$ such that $V(C_1) \cup V(C_2) \subseteq V(C'_1) \cup V(C'_2)$ and $|V(C'_1) \cap V(C'_2)| \geq k$. 
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Without using Chen-Chen-Liu Conjecture, we prove Fouquet-Jolivet Conjecture.
The Key Lemmas

Path Lemma

If $H$ is a subgraph of a $k$-connected graph $G$, and $u, v \in V(G)$, then $G$ has a $u, v$-path $P$ with $\alpha(H - V(P)) \leq \max\{0, \alpha(H) - (k - 1)\}$. 

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If $H$ and $C$ are disjoint subgraphs of a $k$-connected graph $G$, with $C$ being a cycle of length $\geq k$, then $G$ has a cycle $C'$ such that $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.
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Multicycle Lemma
If $G$ is a $k$-connected graph with $\alpha(G) = \alpha$, and $0 \leq l \leq \alpha - k$, then there exists cycles $C_0, \cdots, C_l$ satisfying the following:

1. $\alpha(G - \bigcup_{i=0}^{l} V(C_i)) \leq \alpha - k - l$
2. $|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)| \leq \frac{|V(C_0)|}{k} - 1$ for $a \leq i \leq l$. 

O, West, and Wu: Longest Cycles in $k$-connected Graphs with Given Independence
The outline of the proof of Fouquet-Jolivet Conjecture

Path Lemma $\Rightarrow$
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Path Lemma \implies Cycle Lemma \implies
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Path Lemma $\Rightarrow$ Cycle Lemma $\Rightarrow$

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Multicycle Lemma \(\Rightarrow\) Fouquet-Jolivet Conjecture

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Fouquet-Jolivet Conjecture is true.
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n = |V(C_0)| + \sum_{i=1}^{l} |V(C_i) - \cup_{j=0}^{i-1} V(C_j)| \leq |V(C_0)| + (\alpha - k) \left( \frac{|V(C_0)|}{k} - 1 \right)
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The inequality simplifies to $|V(C_0)| \geq \frac{k(n + \alpha - k)}{\alpha}$.
Kouider’s Theorem

Before proving that Cycle Lemma $\Rightarrow$ Multicycle Lemma, we need to understand Kouider’s Theorem.
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The proof of Fouquet-Jolivet Conjecture

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**Theorem (Kouider 1994)**

If $H$ is a subgraph of a $k$-connected graph $G$, then $G$ has a cycle $C$ such that

$$\alpha(H - V(C)) \leq \max\{0, \alpha(H) - k\}.$$
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Kouider’s Theorem implies the following corollary conjectured by Amar, Fournier, Häggkvist, and Thomassen.
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Kouider’s Theorem implies the following corollary conjectured by Amar, Fournier, Häggkvist, and Thomassen.

**Corollary (Kouider 1994)**

If $G$ is a $k$-connected graph with $\alpha(G) = \alpha$, then $V(G)$ can be covered by $\left\lceil \frac{\alpha}{k} \right\rceil$ cycles.
Cycle Lemma ⇒ Multicycle Lemma

**Cycle Lemma**

If $H$ and $C$ are disjoint subgraphs of a $k$-connected graph $G$, with $C$ being a cycle of length $\geq k$, then $G$ has a cycle $C'$ such that $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$. 
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Note that $|V(C_0)| \geq k$ from (1) for $l = 0$. 
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Note that $|V(C_0)| \geq k$ from (1) for $l = 0$.

By setting $H = G - \bigcup_{i=0}^{l-1} V(C_i)$, $\alpha(H) \leq \alpha - k - (l - 1)$. 
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$|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$. If we add $C'$ into the list, then we have the condition (1).
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Case 1: $|V(C')| \leq |V(C_0)|$
By taking $C_l = C'$, we have

$$|V(C') - \bigcup_{j=0}^{i-1} V(C_j)| \leq |V(C') - V(C_0)| \leq |V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$$
If $\alpha(H) = 0$, then take $C_l = C_0$.
Otherwise, apply Cycle Lemma with $C_0$ as $C$ to get $C'$:
$|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.
If we add $C'$ into the list, then we have the condition (1).

Case 1: $|V(C')| \leq |V(C_0)|$
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Case 2: $|V(C')| > |V(C_0)|$
If $\alpha(H) = 0$, then take $C_l = C_0$.

Otherwise, apply Cycle Lemma with $C_0$ as $C$ to get $C'$:

$|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$. If we add $C'$ into the list, then we have the condition (1).

Case 1: $|V(C')| \leq |V(C_o)|$

By taking $C_l = C'$, we have

$|V(C') - \bigcup_{j=0}^{i-1} V(C_j)| \leq |V(C') - V(C_0)| \leq |V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$

Case 2: $|V(C')| > |V(C_o)|$

Set $C'_o = C'$ and $C'_i = C_{i-1}$ for $1 \leq i \leq l$
If $\alpha(H) = 0$, then take $C_l = C_0$.
Otherwise, apply Cycle Lemma with $C_0$ as $C$ to get $C'$:

- $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

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**Case 2:** $|V(C')| > |V(C_o)|$

Set $C'_o = C'$ and $C'_i = C_{i-1}$ for $1 \leq i \leq l$.

For $i = 1$, $V(C'_1) - V(C'_0) = V(C_0) - V(C')$. 
If $\alpha(H) = 0$, then take $C_l = C_0$. Otherwise, apply Cycle Lemma with $C_0$ as $C$ to get $C'$:

$$|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$. If we add $C'$ into the list, then we have the condition (1).

Case 1: $|V(C')| \leq |V(C_o)|$

By taking $C_l = C'$, we have

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Set $C'_o = C'$ and $C'_i = C_{i-1}$ for $1 \leq i \leq l$

For $i = 1$, $V(C'_1) - V(C'_0) = V(C_0) - V(C')$.

For $i \geq 2$, $|V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j)| \leq \frac{|V(C_o)|}{k} - 1 \leq \frac{|V(C'_0)|}{k} - 1$. 
The outline of the proof of **Fouquet-Jolivet Conjecture**

- Path Lemma $\Rightarrow$ Cycle Lemma $\Rightarrow$

- Multicycle Lemma $\Rightarrow$ **Fouquet-Jolivet Conjecture**
The outline of the proof of **WOW Theorem**

Path Lemma ⇒ Cycle Lemma ⇒

Multicycle Lemma ⇒ WOW Theorem