Edge-connectivity, Eigenvalues, and Matchings in Regular Graphs

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Discrete Mathematics Seminar
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Eigenvalues and Matching Number in Regular Graphs
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In fact, the above graph is the smallest cubic graph without a perfect matching.
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- **How small** can the matching number of an \(n\)-vertex connected cubic graph be? The answer was determined by Biedl et al. Broere et al. gave a formula for the minimum size of a matching in a \((k - 2)\)-edge-connected \(k\)-regular graph with a fixed number of vertices. Henning and Yeo extended this to \(n\)-vertex connected \(k\)-regular graphs.
O and West also gave a short proof of their bound, characterized the extremal graphs, and studied the relationship between the matching number and the number of cut-edges.
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- More generally, how small can the matching number, $\alpha'(G)$ of an $n$-vertex $l$-edge-connected $k$-regular graph $G$ be?
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We prove a lower bound for the minimum size of a maximum matching in an $l$-edge-connected $k$-regular graph with $n$ vertices.
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- More generally, how small can the matching number, $\alpha'(G)$ of an $n$-vertex $l$-edge-connected $k$-regular graph $G$ be?

We prove a lower bound for the minimum size of a maximum matching in an $l$-edge-connected $k$-regular graph with $n$ vertices. Our lower bound includes the earlier special cases when the parameters are set to appropriate values, except in the case $l = 1$, which was resolved by Henning and Yeo.
Let $c(G)$ and $\alpha'(G)$ be the number of cut-edges and the matching number of a graph $G$. 
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**Theorem (Chartrand et.al. 1984)**

If $G$ is an $n$-vertex connected cubic graph, then $\alpha'(G) \geq \frac{n}{2} - \frac{c(G)}{3}$
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From these two theorems, we have $\frac{7n+14}{18}$ as a lower bound for the matching number over $n$-vertex cubic graphs.
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Creation of Balloon

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$B_r$: the unique graph with $2r + 3$ vertices having $2r + 2$ vertices of degree $2r + 1$ and one vertex of degree $2r$. ($B_r = P_3 + rK_2$)
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$B_r$ is the smallest possible balloon in a $(2r + 1)$-regular graph.
Balloons

Let $\mathcal{F}_n$ be the family of connected $(2r + 1)$-regular graphs with $n$ vertices, and let $b(G)$ be the number of balloons in a graph $G$. 
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**Theorem (O and West 2009)**

If $G \in \mathcal{F}_n$, then $b(G) \leq \frac{(2r-1)n+2}{4r^2+4r-2}$, which reduces to $\frac{n+2}{6}$ for cubic graphs. Equality holds infinitely often.
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**Lemma (O and West 2009)**

Let $G$ be an $n$-vertex $(2r+1)$-regular graph, and let $S$ be a subset of $V(G)$. If the number of edges from each odd component of $G - S$ to $S$ is only 1 or at least $2r + 1$, then $\text{def}(S) \leq \frac{2rb(G)}{2r+1}$. 
Theorem (O and West 2009)

If $G \in F_n$, then $c(G) \leq \frac{r(n-2)-2}{2r^2+2r-1} - 1$, which reduces to $\frac{n-7}{3}$ for cubic graphs. Equality holds infinitely often.
Results using Balloons

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Theorem (Henning and Yeo 2007)

If $G \in \mathcal{F}_n$, then $\alpha'(G) \geq \frac{n}{2} - \frac{r}{2} \frac{(2r-1)n+2}{(2r+1)(2r^2+2r-1)}$. 
Construction - $\mathcal{T}_r$ and $\mathcal{H}_r$

- $\mathcal{T}_r$: the family of trees such that every non-leaf vertex has degree $2r + 1$ and all the leaves have the same color in a proper 2-coloring.
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- $\mathcal{T}_r$: the family of trees such that every non-leaf vertex has degree $2r + 1$ and all the leaves have the same color in a proper 2-coloring.

- $\mathcal{H}_r$: the family of $(2r + 1)$-regular graphs obtained from trees in $\mathcal{T}_r$ by identifying each leaf of such a tree with the vertex of degree $2r$ in a copy of $B_r$. 
Properties of graphs in $\mathcal{H}_r$

**Proposition (O and West 2009)**

Let $G$ be an $n$-vertex graph in $\mathcal{H}_r$.

- $n \equiv 4(r + 1)^2 \mod (8r^3 + 12r^2 - 2)$
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Theorem (O and West 2009)

When $n$ is as above, $G$ has the smallest matching number over all connected $(2r + 1)$-regular graphs if and only if $G$ is in $\mathcal{H}_r$. 
Balloons and Total Domination

- A subset $S$ is a **dominating set** in a graph $G$ if every vertex outside $S$ has a neighbor in $S$. 
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- A subset $S$ is a **dominating set** in a graph $G$ if every vertex outside $S$ has a neighbor in $S$.
- A subset $S$ is a **total dominating set** in a graph $G$ if every vertex in $V(G)$ has a neighbor in $S$. The **total domination number** of $G$, denoted $\gamma_t(G)$, is the least size of such a set.
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**Theorem (Henning, Kang, Shan, and Yeo 2008)**

When $G$ is regular with degree at least 3, $\gamma_t(G) \leq \alpha'(G)$. 
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When $G$ is regular with degree at least 3, $\gamma_t(G) \leq \alpha'(G)$.

**Theorem (O and West 2009)**

If $G$ is a connected cubic graph, then $\gamma_t(G) \leq \alpha'(G) - b(G)/6$, unless $b(G) = 3$ and there is only one vertex outside the balloons.
Lower bound

**Theorem (Henning and Yeo 2007)**

If $G$ is a connected $(2r + 1)$-regular graph with $n$ vertices, then

$$
\alpha'(G) \geq \frac{n}{2} - \frac{r}{2} \frac{(2r-1)n+2}{(2r+1)(2r^2+2r-1)}.
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Theorem (O and West 2009+)
If $G$ is a $(2t + 1)$-edge-connected $(2r + 1)$-regular graph with $n$ vertices, then $\alpha'(G) \geq \frac{n}{2} - \left(\frac{r-t}{2(r+1)^2+t}\right)\frac{n}{2}$, and this is sharp for $t \geq 1$. 
Since the degree sum of any graph is even, it follows that every edge cut in a regular graph of even degree has even size, and every edge cut in a regular graph of odd degree that breaks the vertex set into odd-sized sets has odd size.
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Deficiency: \(\text{def}(S) = o(G - S) - |S|\) for \(S \subseteq V(G)\).
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**Tutte’s Theorem:**
\(G\) has a 1-factor iff \(\text{def}(S) \leq 0\) for all \(S\).
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**Tutte’s Theorem** : \(G\) has a 1-factor iff \(\text{def}(S) \leq 0\) for all \(S\).

**Berge-Tutte Formula** : \(\alpha'(G) = \min_{S \subseteq V(G)} \frac{1}{2} (n - \text{def}(S))\).
Proof of Lower bound for Odd Regular Graphs

Proof. Let $S$ be a set with $\alpha'(G) = \frac{1}{2}(n - \text{def}(S))$. 
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Let $c_i$ count the odd components of $G - S$ having $i$ edges to $S$.
Let $c = c(2t+1) + \cdots + c(2r-1)$, and let $c' = o(G - S) - c$. 
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Note that for $(2t + 1) \leq i \leq 2r - 1$, each odd component of $G - S$ having $i$ edges to $S$ has at least $(2r + 3)$ vertices.
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Counting the edges joining \( S \) to odd components of \( G - S \) yields
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(2r + 1)|S| \geq (2r + 1)c' + (2t + 1)c,
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so \( |S| \geq c' + \left(\frac{2t+1}{2r+1}\right)c \).
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Therefore, $n \geq |S| + c(2r + 3) \geq \left(\frac{2r+1}{2r+1}\right)c + c(2r + 3)$,

which implies that $c \leq \left(\frac{2r+1}{4r^2+4r+4+2t}\right)n$. 
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Now, $\text{def}(S) = (c + c') - |S| \leq c - \frac{2t+1}{2r+1}c = \frac{2(r-t)}{2r+1}c \leq \frac{(r-t)n}{2(r+1)^2 + t}$.
Lower Bound for Even Regular Graphs

**Theorem (O and West 2009+)**

If $G$ is a $2t$-edge-connected $2r$-regular graph with $n$ vertices, then

$$\alpha'(G) \geq \frac{n}{2} - \left(\frac{r-t}{2r^2 + r + t}\right)\frac{n}{2}.$$
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Also, for $2t \leq i \leq 2r - 2$, an odd component of $G - S$ having $i$ edges to $S$ has at least $2r + 1$ vertices.
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The same steps as before then lead to

$$\text{def}(S) \leq \frac{(r - t)n}{2r^2 + r + t}.$$
Definitions and Families for Characterization

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- A **$(2r + 1, 2t + 1)$-bullet** is a graph $H$ satisfying the following:
  1. $|V(H)| = 2r + 3$,
  2. $\delta(H) \geq \max\{2r - 2t, r + 1\}$,
  3. $\Delta(H) = 2r + 1$,
  4. $|E(H)| = r + t + 2$, and
  5. every nontrivial cut has at least $2r + 1$ edges.
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- An $(a, b)$-biregular graph is a bipartite graph with partite sets $A$ and $B$ such that vertices in $A$ have degree $a$ and those in $B$ have degree $b$. 
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- Splicing a $B_{2,1}$ into a vertex with degree 3 in $H$:
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- $\mathcal{H}_{r,t} : (2t + 1)$-edge-connected $(2r + 1, 2t + 1)$-biregular graphs
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Definitions and Families for Characterization

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Lemma (O and West 2009+)

If $G \in \mathcal{F}_{r,t}$, then $G$ is $(2t + 1)$-edge-connected and $(2r + 1)$-regular.
Characterization for Odd regular Graphs

Theorem (O and West 2009+)

For $t, r \in \mathbb{N}$ with $t < r$, a $(2t + 1)$-edge-connected $(2r + 1)$-regular graph $G$ achieves equality in the lower bound on $\alpha'(G)$ if and only if it is in $\mathcal{F}_{r,t}$. 
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Hence $|S| = \frac{(2t+1)n}{4r^2+8r+r+2t}$. 

Motivation and Questions
Matching number and Edge-connectivity in Regular Graphs
Eigenvalues and Matching Number in Regular Graphs

Cioaba, O, and West
Edge-connectivity, Eigenvalues and, Matchings in Regular Graphs
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Thus, \( \text{def}(G) \leq |S'| - |S| = \left(\frac{2r+1}{2t+1} - 1\right)|S| = \frac{(r-t)n}{2(r+1)^2+t} \).
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By previous theorem, \( \text{def}(G) \leq \frac{(r-t)n}{2(r+1)^2+t} \); hence equality holds.
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Completion of Proof

Conversely, we want to show that every graph $G$ achieving equality in the bound is in $F_{r,t}$.

To achieve equality in the bound, equality in all inequalities in the computation is required.

This enforces many requirements on the structure of $G$, leading to $G \in F_{r,t}$, which completes the proof.
In order to characterize the even regular graphs achieving equality, we define special families.
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• A \((2r, 2t)\)-bullet is a graph \(H\) satisfying the following:
  1. \(|V(H)| = 2r + 1\),
  2. \(\delta(H) \geq \max\{2r - 2t, r\}\),
  3. \(\Delta(H) = 2r\),
  4. \(|E(H)| = t\), and
  5. every nontrivial cut has at least \(2r\) edges.
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**Theorem (O and West 2009+)**

For \(t, r \in \mathbb{N}\) with \(t < r\), a \(2t\)-edge-connected \(2r\)-regular graph \(G\) achieves equality in the bound of Theorem if and only if it is in \(\mathcal{F}_{r,t}'\).
Infinitely Many Graphs Achieving Equality

One may wonder whether there are infinitely many graphs in the family $\mathcal{F}_{r,t}$. (Similarly for $\mathcal{F}'_{r,t}$).
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![Graphs](image-url)
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```
- Let $B_r = P_3 + rK_2$.
- For $0 \leq t < r$, let $B_{r,t}$ be a graph obtained from $B_r$ by deleting a matching of size $t$ whose elements are not incident to degree $2r$.
```
Lemma

For $0 \leq t < r$, the edge-connectivity of the graph $B_{r,t}$ is $2r$. 
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Lemma
Let $H$ be a $(a, b)$-biregular graph with edge-connectivity $a$. For $a = 2t + 1$ and $b = 2r + 1$, splicing $B_{r,t}$ into each vertex having degree $2t + 1$ in $H$ preserves $(2t + 1)$-edge-connectedness.
Construction of $G_k$

To show that equality holds infinitely often, we need to build an infinite family of such $H$. 

Cioaba, O, and West
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Cyclic construction using two copies of $K_{3,5}$

We put $k$ copies of $K_{a,b}$ around a circle and modify them to construct $G_k$ by deleting matchings of size $a$ and reconnecting the deficient vertices via new matchings to the next “piece”.
Edge Connectivity of $G_k$

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**Lemma**

For $a \geq b$, if a graph $H$ is the graph obtained from $K_{a,b}$ by deleting a matching of size $a$, then $\kappa'(H) = b - 1$. 
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The edge-connectivity of $G_k$ is equal to $a$.

For $a = 2r + 1$ and $b = 2t + 1$, we obtain $G_k \in \mathcal{H}_{r,t}$. 
In 2005, Brouwer and Haemers gave an eigenvalue condition for existence of a perfect matching in regular graphs.
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**Theorem (Brower and Haemers 2005)**

A $2n$-vertex connected $d$-regular graph $G$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n}$, satisfying

$$\lambda_3 \leq \begin{cases} 
    d - 1 + \frac{3}{d+1} & \text{if } d \text{ is even,} \\
    d - 1 + \frac{3}{d+2} & \text{if } d \text{ is odd,}
\end{cases}$$

has a perfect matching.
In 2009, Cioaba, Gregory, and Haemers found a best upper bound (in terms of $d$) on the third largest eigenvalue that is sufficient to guarantee that $G$ has a perfect (or near perfect) matching.
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Let $\mathcal{H}(d)$ denote the class of all connected irregular graphs with maximum degree $d$, odd order $n$, and size $e$ with $2e \geq dn - d + 2$. Define $\rho(d)$ to be the minimum of the spectral radii of the graphs $H$ in $\mathcal{H}(d)$:

$$\rho(d) := \min_{H \in \mathcal{H}(d)} \lambda_1(H).$$
Theorem (Cioaba, Gregory, and Haemers 2009)

If $G$ is a connected $d$-regular graph of order $n$ such that

$$\lambda_3 < \rho(d),$$

then $\alpha'(G) = \left\lfloor \frac{n}{2} \right\rfloor$. 
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Theorem (Cioaba, Gregory and Haemers 2009)

If $\theta$ is the largest root of $x^3 - x^2 - 6x + 2 = 0$, then
\[
\rho(d) = \begin{cases} 
\theta = 2.85577... & \text{if } d = 3, \\
\frac{1}{2}(d - 2 + \sqrt{d^2 + 12}) & \text{if } d \geq 4 \text{ is even}, \\
\frac{1}{2}(d - 3 + \sqrt{(d + 1)^2 + 16}) & \text{if } d \geq 5 \text{ is odd}. 
\end{cases}
\]
Useful Definitions and Theorems for Eigenvalues

• A partition $\pi$ of $V(G)$ with cells $C_1, \ldots, C_r$ is equitable if the number of neighbors in $C_j$ of a vertex $u$ in $C_i$ is a constant $b_{ij}$, independent of $u$. 
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**Lemma (Interlacing Theorem)**

Let $A$ be a real symmetric $n \times n$ matrix, and let $B$ be a principal submatrix of $A$ with order $m \times m$. Then, for $1 \leq i \leq m$,

$$\theta_{n-m+1} \leq \theta_i(B) \leq \theta_i(A).$$
Lemma (Cioaba and O 2009+)

The spectral radius of $B_r$, denoted $\tau_r$, equals the largest root of the equation

$$x^3 - (2r - 1)x^2 - 2(2r + 1)x + 2r = 0.$$
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Proof. Partition the vertex set $V(B_r)$ into three parts: the vertex of degree $2r$, its $2r$ neighbors and the remaining two vertices.
Balloons and Eigenvalues

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This partition is equitable. Form its quotient matrix.
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Since the partition is equitable, the spectral radius of $B_r$ equals the largest root of the characteristic polynomial of the quotient matrix.
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If $B$ is a balloon in a $(2r + 1)$-regular graph, then $\lambda_1(B) \geq \tau_r$, with equality iff $B = B_r$. 
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Proof. Assume that $n \geq 2r + 5$, where $n = |V(B)|$. We want to show that $\lambda_1(B) > \tau_r$. 
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Clearly, the average degree of $B$ is $\frac{(n-1)(2r+1)+2r}{n} = 2r + 1 - \frac{1}{n}$. 
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If $B$ is a balloon in a $(2r+1)$-regular graph, then $\lambda_1(B) \geq \varpi$, with equality iff $B = B_r$.

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$$\lambda_1(B) > 2r + 1 - \frac{1}{n} \geq 2r + 1 - \frac{1}{2r + 5} > \varpi$$

for $r \geq 2$, or $r = 1$ and $n \geq 9$. 
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for $r \geq 2$, or $r = 1$ and $n \geq 9$. If $r = 1$ and $n = 7$, then there are only four graphs.
Balloons and Eigenvalues

Theorem (Cioaba and O 2009+)

If $G$ is a regular graph with odd degree, and $\lambda_k(G) < \tau_r$, then $b(G) \leq k - 1$. 
Balloons and Eigenvalues

**Theorem (Cioaba and O 2009+)**

If $G$ is a regular graph with odd degree, and $\lambda_k(G) < \tau_r$, then $b(G) \leq k - 1$.

Proof. If $G$ has at least $k$ balloons $H_1, \ldots, H_k$, then

$$\lambda_k(G) \geq \lambda_k(H_1 \cup \cdots \cup H_k) \geq \min_{1 \leq i \leq k} (\lambda_1(H_i)) \geq \tau_r$$
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$$\lambda_k(G) \geq \lambda_k(H_1 \cup \cdots \cup H_k) \geq \min_{1 \leq i \leq k} (\lambda_1(H_i)) \geq \tau_r$$

Corollary

If $G$ is an $n$-vertex $(2r + 1)$-regular graph having the smallest matching number, then $\lambda_{\left\lfloor \frac{(2r-1)n+2}{4r^2+4r-2} \right\rfloor}(G) \geq \tau_r$. 
Main Result

**Theorem (Cioaba and O 2009+)**

Let \( p \geq 3 \) be an integer. If \( G \) is a \( t \)-edge-connected \( d \)-regular graph such that \( \lambda_p(G) < \rho(d) \), then

\[
\alpha'(G) > \begin{cases} 
\frac{n-p + \lfloor \frac{tp}{d} \rfloor}{2} & \text{when } d \equiv t \pmod{2} \\
\frac{n-p + \lfloor \frac{(t+1)p}{d} \rfloor}{2} & \text{when } d \equiv t + 1 \pmod{2}.
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**Corollary (Cioaba, Gregory, and Haemaers 2009 JCTB)**

If $G$ is a connected $d$-regular graph of order $n$ such that $\lambda_3 < \rho(d)$, then $\alpha'(G) > \frac{n-2}{2}.$
Corollary (Cioaba and O 2009+)

Let $d \geq 3$ and $t \leq d - 2$ be two integers. If $G$ is a $t$-edge-connected $d$-regular graph of order $n$ such that

$$\rho(d) > \begin{cases} 
\lambda \left\lfloor \frac{2d}{d-t} \right\rfloor & \text{if } d \equiv t \pmod{2} \\
\lambda \left\lfloor \frac{2d}{d-(t+1)} \right\rfloor & \text{if } d \equiv t + 1 \pmod{2}
\end{cases}$$

then \( \alpha'(G) = \left\lfloor \frac{n}{2} \right\rfloor \).
Parity Lemma

If $G$ is a $t$-edge-connected $d$-regular graph, and $S$ is a nonempty proper subset of $V(G)$ with odd order, then

$$|S, G - S| \geq \begin{cases} 
  t & \text{if } d \equiv t \pmod{2} \\
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\end{cases}$$

Lemma (Cioaba and O 2009+)

Let $G$ be a $t$-edge-connected $d$-regular graph with $n$ vertices. If $\alpha'(G) \leq \frac{n-a}{2}$, then

$$\rho(d) \leq \begin{cases} 
  \lambda \left[ \frac{ad}{d-t} \right] & \text{if } d \equiv t \pmod{2}, \\
  \lambda \left[ \frac{ad}{d-(t+1)} \right] & \text{if } d \equiv t + 1 \pmod{2}.
\end{cases}$$ (1)
Proof of Lemma

Let $S$ be a subset of $G$ with maximum deficiency.
Proof of Lemma

Let $S$ be a subset of $G$ with maximum deficiency. Let $O_1, O_2, ..., O_q$ be all the odd components of $G \setminus S$, where $o(G \setminus S) = q$. 
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Let $S$ be a subset of $G$ with maximum deficiency. Let $O_1, O_2, \ldots, O_q$ be all the odd components of $G \setminus S$, where $o(G \setminus S) = q$. Because $\alpha'(G)$ is at most $\frac{n-a}{2}$, we get $q \geq |S| + a$. 
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Motivation and Questions

Matching number and Edge-connectivity in Regular Graphs

Eigenvalues and Matching Number in Regular Graphs

Proof of Lemma

Let $S$ be a subset of $G$ with maximum deficiency. Let $O_1, O_2, ..., O_q$ be all the odd components of $G \setminus S$, where $\alpha(G \setminus S) = q$. Because $\alpha'(G)$ is at most $\frac{n-a}{2}$, we get $q \geq |S| + a$. Let $n_i = |O_i|, e_i = |E(O_i)|$ and $t_i = |[S, O_i]|$ for $1 \leq i \leq q$. Because $G$ is $t$-edge-connected, by Parity Lemma, $t_i \geq t$ when $d \equiv t \pmod{2}$, and $t_i \geq t + 1$ when $d \equiv t + 1 \pmod{2}$. 

Cioaba, O, and West

Edge-connectivity, Eigenvalues and, Matchings in Regular Graphs
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Let $S$ be a subset of $G$ with maximum deficiency. Let $O_1, O_2, ..., O_q$ be all the odd components of $G \setminus S$, where $o(G \setminus S) = q$. Because $\alpha'(G)$ is at most $\frac{n-a}{2}$, we get $q \geq |S| + a$. Let $n_i = |O_i|$, $e_i = |E(O_i)|$ and $t_i = |[S, O_i]|$ for $1 \leq i \leq q$. Because $G$ is $t$-edge-connected, by Parity Lemma, $t_i \geq t$ when $d \equiv t \pmod{2}$, and $t_i \geq t + 1$ when $d \equiv t + 1 \pmod{2}$. We will prove the lemma for $d \equiv t \pmod{2}$, the proof of the other case being similar. By counting the number of edges between $S$ and $G \setminus S$, we have $d|S| \geq |[S, G \setminus S]| \geq \sum_{i=1}^{q} t_i \geq qt \geq (|S| + a)t$. Thus, $(d - t)|S| \geq at$, which implies that $|S| \geq \frac{at}{d-t}$. Because $q \geq |S| + a$, we obtain $q \geq \frac{at}{d-t} + a = \frac{ad}{d-t}$.
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Proof of Lemma

Let $S$ be a subset of $G$ with maximum deficiency. Let $O_1, O_2, \ldots, O_q$ be all the odd components of $G \setminus S$, where $o(G \setminus S) = q$. Because $\alpha'(G)$ is at most $\frac{n-a}{2}$, we get $q \geq |S| + a$.

Let $n_i = |O_i|$, $e_i = |E(O_i)|$ and $t_i = |[S, O_i]|$ for $1 \leq i \leq q$. Because $G$ is $t$-edge-connected, by Parity Lemma, $t_i \geq t$ when $d \equiv t \pmod{2}$, and $t_i \geq t + 1$ when $d \equiv t + 1 \pmod{2}$. We will prove the lemma for $d \equiv t \pmod{2}$, the proof of the other case being similar. By counting the number of edges between $S$ and $G \setminus S$, we have $d|S| \geq |[S, G \setminus S]| \geq \sum_{i=1}^{q} t_i \geq qt \geq (|S| + a)t$.

Thus, $(d - t)|S| \geq at$, which implies that $|S| \geq \frac{at}{d-t}$. Because $q \geq |S| + a$, we obtain $q \geq \frac{at}{d-t} + a = \frac{ad}{d-t}$. Thus, $q \geq \left\lceil \frac{ad}{d-t} \right\rceil$. We claim that there are at least $\left\lceil \frac{ad}{d-t} \right\rceil$ $t_i$'s such that $t_i < d$. 
Completion of Proof

Otherwise, there are at most \(\left\lfloor \frac{ad}{d-t} \right\rfloor - 1\) \(t_i\)'s such that \(t_i < d\) which means there are at least \(q - \left(\left\lfloor \frac{ad}{d-t} \right\rfloor - 1\right)\) \(t_i\)'s such that \(t_i \geq d\).
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Otherwise, there are at most \( \left\lfloor \frac{ad}{d-t} \right\rfloor - 1 \) \( t_i \)'s such that \( t_i < d \) which means there are at least \( q - \left( \left\lfloor \frac{ad}{d-t} \right\rfloor - 1 \right) \) \( t_i \)'s such that \( t_i \geq d \). Because \( G \) is \( t \)-edge-connected, we know \( t_i \geq t \) for each \( 1 \leq i \leq q \).
Completion of Proof

Otherwise, there are at most \(\left\lfloor \frac{ad}{d-t} \right\rfloor - 1\) \(t_i\)'s such that \(t_i < d\) which means there are at least \(q - \left(\left\lfloor \frac{ad}{d-t} \right\rfloor - 1\right)\) \(t_i\)'s such that \(t_i \geq d\). Because \(G\) is \(t\)-edge-connected, we know \(t_i \geq t\) for each \(1 \leq i \leq q\). These facts imply

\[
d|S| \geq \sum_{i=1}^{q} t_i \geq d \left[q - \left(\left\lfloor \frac{ad}{d-t} \right\rfloor - 1\right)\right] + t \left(\left\lfloor \frac{ad}{d-t} \right\rfloor - 1\right)
\]

\[
= dq - (d - t) \left(\left\lfloor \frac{ad}{d-t} \right\rfloor - 1\right)
\]

\[
> dq - (d - t) \frac{ad}{d-t} = d(q - a)
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which is contradiction.
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Otherwise, there are at most \(\left\lfloor \frac{ad}{d-t} \right\rfloor - 1\) \(t_i\)'s such that \(t_i < d\) which means there are at least \(q - \left(\left\lfloor \frac{ad}{d-t} \right\rfloor - 1\right)\) \(t_i\)'s such that \(t_i \geq d\).

Because \(G\) is \(t\)-edge-connected, we know \(t_i \geq t\) for each \(1 \leq i \leq q\). These facts imply

\[
d|S| \geq \sum_{i=1}^{q} t_i \geq d \left( q - \left(\left\lfloor \frac{ad}{d-t} \right\rfloor - 1\right) \right) + t \left(\left\lfloor \frac{ad}{d-t} \right\rfloor - 1\right)
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\]

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\]

\[
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\]

which is contradiction. Here we used the inequality \(x > \left\lfloor x \right\rfloor - 1\) for any real number \(x\).
Completion of Proof

Let \( p = \left\lceil \frac{ad}{d-t} \right\rceil \).
Completion of Proof

Let \( p = \left\lceil \frac{ad}{d-t} \right\rceil \). We assume that \( t_i < d \) for each \( 1 \leq i \leq p \).
Let $p = \lceil \frac{ad}{d-t} \rceil$. We assume that $t_i < d$ for each $1 \leq i \leq p$. By Theorem, we obtain $\lambda_1(G[O_i]) \geq \rho(d)$ for each $1 \leq i \leq p$. 
Let $p = \lceil \frac{ad}{d-t} \rceil$. We assume that $t_i < d$ for each $1 \leq i \leq p$. By Theorem, we obtain $\lambda_1(G[O_i]) \geq \rho(d)$ for each $1 \leq i \leq p$. This and eigenvalue interlacing imply

$$
\lambda_p(G) \geq \lambda_p(G[O_1 \cup \cdots \cup O_p]) \geq \min_{1 \leq i \leq p} \lambda_1(G[O_i]) \geq \rho(d)
$$

which finishes the proof.