Generalized Balloons and Chinese Postman Problems in Regular Graphs

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1) **How many cut-edges** can a connected \((2r + 1)\)-regular graph with \(n\) vertices have?
2) **How small** can the **matching number** \(\alpha'(G)\) be in a connected \((2r + 1)\)-regular graph with \(n\) vertices?
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2) How small can the **matching number** \(\alpha'(G)\) be in a connected \((2r + 1)\)-regular graph with \(n\) vertices?
3) Can we **characterize when equality holds**? (i.e. when is the matching number minimized?)
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2) How small can the matching number \(\alpha'(G)\) be in a connected \((2r + 1)\)-regular graph with \(n\) vertices?
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4) Are there other applications of balloons?
Definitions

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Definitions

- Graphs in which every vertex has degree 3 are **cubic graphs**.
- The **matching number** $\alpha'(G)$ of a graph is the maximum size of a matching in it.
- Let $c(G)$ denote the number of cut-edges in a graph $G$.
- Let $b(G)$ denote the number of balloons in a graph $G$. 

Tools - Balloons and $B_r$

- A balloon in a graph $G$ is a maximal 2-edge-connected subgraph of $G$ that is incident to exactly one cut-edge of $G$. 

$B_1$ and a cubic graph with two balloons
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- $B_r$: the unique graph with $2r + 3$ vertices having $2r + 2$ vertices of degree $2r + 1$ and one vertex of degree $2r$. ($B_r$ is the complement of $P_3 + rK_2$)
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$B_r$ is the smallest possible balloon in a $(2r + 1)$-regular graph.
$\mathcal{F}_n$: the family of connected $(2r + 1)$-regular graphs with $n$ vertices.
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**Theorem (O and West 2009+)**

If $G \in \mathcal{F}_n$, then $c(G) \leq \frac{r(n-2)-2}{2r^2+2r-1} - 1$ cut-edges, which reduces to $\frac{n-7}{3}$ for cubic graphs. Equality holds infinitely often.
**Previous Results**

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**Theorem (Henning and Yeo 2007)**

If \( G \in \mathcal{F}_n \), then \( \alpha'(G) \geq \frac{n}{2} - \frac{r(2r-1)n+2}{2(2r+1)(2r^2+2r-1)} \).
Construction - $\mathcal{T}_r$ and $\mathcal{H}_r$

$\mathcal{T}_r$ : the family of trees such that every non-leaf vertex has degree $2r + 1$ and all the leaves have the same color in a proper 2-coloring.
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- $\mathcal{T}_r$: the family of trees such that every non-leaf vertex has degree $2r + 1$ and all the leaves have the same color in a proper 2-coloring.

- $\mathcal{H}_r$: the family of $(2r + 1)$-regular graphs obtained from trees in $\mathcal{T}_r$ by identifying each leaf of such a tree with the vertex of degree $2r$ in a copy of $B_r$. 

\[ \text{a graph in } \mathcal{T}_1 \]

\[ \text{a graph in } \mathcal{H}_1 \]
Properties of graphs in $\mathcal{H}_r$

**Proposition (O and West 2009+)**

Let $G$ be an $n$-vertex graph in $\mathcal{H}_r$.

- $n \equiv 4(r + 1)^2 \mod (8r^3 + 12r^2 - 2)$
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Theorem (O and West 2009+)

When $n$ is as above, $G$ has the smallest matching number over all connected $(2r + 1)$-regular graphs if and only if $G$ is in $\mathcal{H}_r$. 

O and West

Generalized Balloons and Chinese Postman Problems
Balloons and Total Domination

- A subset $S$ is a **dominating set** in a graph $G$ if every vertex outside $S$ has a neighbor in $S$. 
Balloons and Total Domination

- A subset $S$ is a **dominating set** in a graph $G$ if every vertex outside $S$ has a neighbor in $S$.
- A subset $S$ is a **total dominating set** in a graph $G$ if every vertex in $V(G)$ has a neighbor in $S$. The **total domination number** of $G$, denoted $\gamma_t(G)$, is the least size of such a set.
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**Theorem (Henning, Kang, Shan, and Yeo 2008)**

When $G$ is regular with degree at least 3, $\gamma_t(G) \leq \alpha'(G)$. 
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**Theorem (Henning, Kang, Shan, and Yeo 2008)**

When $G$ is regular with degree at least 3, $\gamma_t(G) \leq \alpha'(G)$.

**Theorem (O and West 2009+)**

If $G$ is a connected cubic graph, then $\gamma_t(G) \leq \alpha'(G) - b(G)/6$, unless $b(G) = 3$ and there is only one vertex outside the balloons.
What is the smallest matching number for the family of $t$-edge-connected $(2r + 1)$-regular graphs with $n$ vertices? (Also for $2r$-regular graphs.)
New Questions

- What is the smallest matching number for the family of $t$-edge-connected $(2r + 1)$-regular graphs with $n$ vertices? (Also for $2r$-regular graphs.)

- What is the best upper bound for the length of Chinese postman tours in $t$-edge-connected $(2r + 1)$-regular graphs with $n$ vertices?
Smallest Matching Number and Edge-Connectivity

**Theorem (O and West 2009+)**

If $G$ is a $(2t + 1)$-edge-connected $(2r + 1)$-regular graph with $n$ vertices, then $\alpha'(G) \geq \frac{n}{2} - \left(\frac{r-t}{2(r+1)^2+t}\right)\frac{n}{2}$, and this is sharp for $t \geq 1$. 
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If $G$ is a $(2t + 1)$-edge-connected $(2r + 1)$-regular graph with $n$ vertices, then $\alpha'(G) \geq \frac{n}{2} - \left(\frac{r-t}{2(r+1)^2+t}\right)\frac{n}{2}$, and this is sharp for $t \geq 1$.

Proof. Let $S$ be a set with maximum deficiency. Thus, $\alpha'(G) = \frac{1}{2}(n - \text{def}(S))$, where $\text{def}(S) = o(G - S) - |S|$. 
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Let $c_i$ count the odd components of $G - S$ having $i$ edges to $S$. Let $c = c_{(2t+1)} + \ldots + c_{(2r-1)}$, and let $c' = o(G - S) - c$. 
Smallest Matching Number and Edge-Connectivity

**Theorem (O and West 2009+)**

If \( G \) is a \((2t + 1)\)-edge-connected \((2r + 1)\)-regular graph with \( n \) vertices, then \( \alpha'(G) \geq \frac{n}{2} - \left(\frac{r-t}{2(r+1)^2 + t}\right)\frac{n}{2} \), and this is sharp for \( t \geq 1 \).

Proof. Let \( S \) be a set with maximum deficiency. Thus, \( \alpha'(G) = \frac{1}{2}(n - \text{def}(S)) \), where \( \text{def}(S) = o(G - S) - |S| \).

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Note that for \((2t + 1) \leq i \leq 2r - 1\), each odd component of \( G - S \) having \( i \) edges to \( S \) has at least \((2r + 3)\) vertices. (Otherwise, each vertex of \( G \) has a neighbor outside.)
Completion of Proof

Counting the edges joining $S$ to odd components of $G - S$ yields

$$(2r + 1)|S| \geq (2r + 1)c' + (2t + 1)c.$$
Completion of Proof

Counting the edges joining $S$ to odd components of $G - S$ yields

$$(2r + 1)|S| \geq (2r + 1)c' + (2t + 1)c.$$ 

Hence $|S| \geq c' + \left(\frac{2t+1}{2r+1}\right)c \geq \left(\frac{2t+1}{2r+1}\right)c.$
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Therefore, 
$$n \geq |S| + c(2r + 3) \geq \left(\frac{2t+1}{2r+1}\right)c + c(2r + 3),$$ 
which implies that 
$$c \leq \left(\frac{2r+1}{4r^2+4r+4+2t}\right)n.$$ 
Now, we compute...
Completion of Proof

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Therefore,

$$n \geq |S| + c(2r + 3) \geq \left(\frac{2t+1}{2r+1}\right)c + c(2r + 3),$$ 

which implies that

$$c \leq \left(\frac{2r+1}{4r^2+4r+4+2t}\right)n.$$ 

Now, we compute

$$\text{def}(S) = (c + c') - |S| \leq c - \frac{2t + 1}{2r + 1}c = \frac{2(r - t)}{2r + 1}c,$$

$$\leq \frac{2(r - t)}{2r + 1} \left(\frac{2r + 1}{4r^2 + 4r + 4 + 2t}\right)n = \frac{(r - t)n}{2(r + 1)^2 + t}.$$
Theorem (O and West 2009+)

If $G$ is a $2t$-edge-connected $(2r + 1)$-regular graph with $n$ vertices, then again $\alpha'(G) \geq \frac{n}{2} - \left(\frac{r-t}{2(r+1)^2+t}\right) \frac{n}{2}$, and this is sharp.

(same bound as when $\kappa'(G) \geq 2t + 1$)
Bounds for Other Cases

Theorem (O and West 2009+)

If $G$ is a $2t$-edge-connected $(2r + 1)$-regular graph with $n$ vertices, then again $\alpha'(G) \geq \frac{n}{2} - \left(\frac{r-t}{2(r+1)^2+t}\right)\frac{n}{2}$, and this is sharp.

(same bound as when $\kappa'(G) \geq 2t + 1$)

Theorem (O and West 2009+)

If $G$ is a $(2t-1)$-edge-connected $2r$-regular graph with $n$ vertices, then $\alpha'(G) \geq \frac{n}{2} - \left(\frac{r-t}{2r^2+r+t}\right)\frac{n}{2}$, and this inequality is sharp even when $G$ is $2t$-edge-connected.
Sharpness for Odd Regular Graphs - $B_{r,t}$ and $G_{r,t}$

- Let $B_{r,t}$ be a graph obtained from the graph $B_r$ by deleting a matching with $t$ edges in $B_r$. 

$B_{2,1}$ a (5, 3)-biregular bigraph $H$

$G_{2,1}$
Let $B_{r,t}$ be a graph obtained from the graph $B_r$ by deleting a matching with $t$ edges in $B_r$.

To make a $(2t + 1)$-edge-connected $(2r + 1)$-regular $G_{r,t}$, replace each vertex of $T$ in a $(2t + 1)$-edge-connected $(2r + 1, 2t + 1)$-biregular $(S, T)$-bigraph $H$ with a copy of $B_{r,t}$. 
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Note: $|S| = q(2t + 1)$ and $|T| = q(2r + 1)$ for some $q \in \mathbb{Q}$. 
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Matching Number of $G_{r,t}$

**Proposition (O and West 2009+)**

For $0 \leq t \leq r$, $\alpha'(G_{r,t}) = \frac{n}{2} - \left(\frac{r-t}{2(r+1)^2+t}\right)\frac{n}{2}$,

where $n = q(2t + 1) + q(2r + 1)(2r + 3) = q \left( (2r + 2)^2 + 2t \right)$.
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where $n = q(2t + 1) + q(2r + 1)(2r + 3) = q \left((2r + 2)^2 + 2t\right)$.

Proof. By the previous theorem, $\alpha'(G_{r,t}) \geq \frac{n}{2} - \left(\frac{r-t}{2(r+1)^2+t}\right)\frac{n}{2}$. 
For $0 \leq t \leq r$, $\alpha'(G_{r,t}) = \frac{n}{2} - \left(\frac{r-t}{2(r+1)^2+t}\right)\frac{n}{2}$, where $n = q(2t+1) + q(2r+1)(2r+3) = q((2r+2)^2 + 2t))$.

Proof. By the previous theorem, $\alpha'(G_{r,t}) \geq \frac{n}{2} - \left(\frac{r-t}{2(r+1)^2+t}\right)\frac{n}{2}$.

Recall that $|S| = (2t+1)q$. Thus,

$$\text{def}(S) = o(G - S) - |S| = q(2r + 1) - q(2t + 1)$$

$$= 2q(r - t) = 2\frac{n}{(2r + 2)^2 + 2t}(r - t) = \frac{(r - t)n}{2(r + 1)^2 + t}.$$
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Edge-Connectivity of \( B_{r,t} \)

**Lemma**

For \( 0 \leq t \leq r \), the edge-connectivity of the graph \( B_{r,t} \) is \( 2r \).
Edge-Connectivity of $B_{r,t}$

**Lemma**
For $0 \leq t \leq r$, the edge-connectivity of the graph $B_{r,t}$ is $2r$.

Proof. (Elementary exercise) If a graph $G$ is connected, and $\frac{n}{2} \leq \delta(G) \leq n$, then $\kappa'(G) = \delta(G)$, and the above is a special case.
Edge-Connectivity of $B_{r,t}$

**Lemma**

For $0 \leq t \leq r$, the edge-connectivity of the graph $B_{r,t}$ is $2r$.

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$$\frac{n}{2} \leq \delta(G) \leq n,$

then $\kappa'(G) = \delta(G)$, and the above is a special case.

**Lemma**

Iteratively replacing a vertex of $T$ with $B_{r,t}$ in $H$ preserves $(2t + 1)$-edge-connectedness.
To show that equality holds infinitely often, we need to build an infinite family of such $H$.

Cyclic construction using two copies of $K_{3,5}$, and $B_{2,1,2}$

We are gonna put a bunch of copies of $K_{2t+1,2r+1}$ around a circle and modify them to construct $H$. 
Edge Connectivity of $B_{r,t,k}$

**Lemma**

For $a \geq b$, if a graph $H$ is the graph obtained from $K_{a,b}$ by deleting a matching of size $b$, then $\kappa'(H) = b - 1$. 
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For $a \geq b$, if a graph $H$ is the graph obtained from $K_{a,b}$ by deleting a matching of size $b$, then $\kappa'(H) = b - 1$.

Proof. (Elementary exercise) If a graph $G$ is a bipartite graph with diameter at most 3, then $\kappa'(G) = \delta(G)$, and the above is a special case.
Edge Connectivity of $B_{r,t,k}$

**Lemma**

For $a \geq b$, if a graph $H$ is the graph obtained from $K_{a,b}$ by deleting a matching of size $b$, then $\kappa'(H) = b - 1$.

**Proof.** (Elementary exercise) If a graph $G$ is a bipartite graph with diameter at most 3, then $\kappa'(G) = \delta(G)$, and the above is a special case.

**Theorem (O and West 2009+)**

For $0 \leq t \leq r$, $\kappa'(B_{r,t,k}) = 2t + 1$
Sharpness for Even Regular Graphs: $B'_{r,t}$ and $G'_{r,t}$

Let $B'_{r,t}$ be the graph obtained from $K_{2r+1}$ by deleting a matching of with $t$ edges.

To make a $2t$-edge-connected $2r$-regular graph $G'_{r,t}$, replace each vertex of $T$ in $2t$-edge-connected $(2r,2t)$-biregular $(S,T)$-bipartite $H'$ with a copy of $B'_{r,t}$.

Similarly, we can have an infinite family of $2t$-edge-connected $2r$-regular graphs like in previous steps.
A **Chinese Postman tour** in a connected graph $G$ is a shortest closed walk traversing all edges in $G$. 
Chinese Postman Problem

A **Chinese Postman tour** in a connected graph $G$ is a shortest closed walk traversing all edges in $G$. Let $e_P(G)$ be the number of edges in it.
A **Chinese Postman tour** in a connected graph $G$ is a shortest closed walk traversing all edges in $G$. Let $e_P(G)$ be the number of edges in it.

A **parity subgraph** in a graph $G$ is a spanning subgraph $H$ of $G$ such that $d_G(v) \equiv d_H(v) \pmod{2}$ for every vertex $v$ in $G$. 
A **Chinese Postman tour** in a connected graph $G$ is a shortest closed walk traversing all edges in $G$.

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A **parity subgraph** in a graph $G$ is a spanning subgraph $H$ of $G$ such that $d_G(v) \equiv d_H(v) \pmod{2}$ for every vertex $v$ in $G$.

Let $p(G)$, the **parity number** of $G$, be the minimum number of edges in a parity subgraph of $G$. 
A **Chinese Postman tour** in a connected graph $G$ is a shortest closed walk traversing all edges in $G$.

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Let $p(G)$, the **parity number** of $G$, be the minimum number of edges in a parity subgraph of $G$.

Note that $e_P(G) = |E(G)| + p(G)$. In view of many applications of the Chinese Postman problem, it is natural to ask for the value of $e_P(G)$, or equivalently, the value of $p(G)$.
Construction of $\mathcal{H}_r$

Let $T'_r$ be the family of trees such that every non-leaf vertex has degree $2r + 1$. 

- a graph in $T'_1$

- a graph in $\mathcal{H}_1'$
Construction of $\mathcal{H}_r'$

- Let $\mathcal{T}_r'$ be the family of trees such that every non-leaf vertex has degree $2r + 1$.
- Let $\mathcal{H}_r'$ be the family of $(2r + 1)$-regular graphs obtained from trees in $\mathcal{T}_r'$ by identifying each leaf of such a tree with the neck in a copy of $B_r$. 
Construction of $\mathcal{H}_r'$

Let $\mathcal{T}_r'$ be the family of trees such that every non-leaf vertex has degree $2r + 1$.

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For $r = 1$, we will show the graph in $\mathcal{H}_r'$ have the largest value of $p(G)$ among $n$-vertex cubic graphs.
Parity Number of $\mathcal{H}'_r$

**Proposition**

Let $p_r = 2r^2 + 2r - 1$. For any $n$-vertex graph $G$ in $\mathcal{H}'_r$,

\[
b(G) = \frac{(2r-1)n+2}{2p_r}, \quad c(G) = \frac{r(n-2)+2}{p_r} - 1.
\]
**Parity Number of $\mathcal{H}'_r$**

**Proposition**

Let $p_r = 2r^2 + 2r - 1$. For any $n$-vertex graph $G$ in $\mathcal{H}'_r$, 

\[ b(G) = \frac{(2r-1)n+2}{2pr}, \quad c(G) = \frac{r(n-2)-2}{pr} - 1. \]

**Lemma**

If $G$ is regular of odd degree, then every cut-edge is in every parity subgraph.
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If $G$ is regular of odd degree, then every cut-edge is in every parity subgraph.

**Corollary**

If $G$ is a graph in $\mathcal{H}_r'$, and $T$ is the tree obtained by shrinking each $B_r$ in $G$ to one vertex, then every parity subgraph of $G$ contains $T$. 
Parity Number of $\mathcal{H}_r$

Theorem (O and West 2009+)

If $G$ is in $\mathcal{H}_r$, then $p(G) = \frac{(2r^2 + 3r - 1)n - 2(r+1)}{4r^2 + 4r - 2} - 1$, which reduces to $\frac{2n - 5}{3}$ for cubic graphs.
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O and West  
Generalized Balloons and Chinese Postman Problems
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$$p(G) = c(G) + (r + 1)b(G) = \frac{r(n - 2) - 2}{p_r} - 1 + (r + 1)\frac{(2r - 1)n + 2}{2p_r}$$

$$= \frac{2r(n - 2) - 4 + (r + 1)(2r - 1)n + 2}{2p_r} - 1 = \frac{(2r^2 + 3r - 1)n - 2(r + 1)}{4r^2 + 4r - 2} - 1.$$
Definitions and Remarks for the upper bound

- An $r$-graph is an $r$-regular multigraph $G$ on an even number of vertices with the property that every edge-cut which separates $V(G)$ into two sets of odd cardinality has size at least $r$. 

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- More generally, if $G$ is an $(r - 1)$-edge-connected $r$-regular multigraph with even order, then $G$ is $r$-graph.
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Application of Edmonds’ Theorem

**Lemma (Edmonds 1965)**

If $G$ is an $r$-graph, then there is an integer $p$ and a family $\mathcal{M}$ of perfect matchings such that each edge of $G$ is contained in precisely $p$ members of $\mathcal{M}$. (The members of $\mathcal{M}$ need not be distinct.)
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**Lemma (O and West 2009+)**
Let $G$ be a $2r$-edge-connected $(2r + 1)$-regular multigraph. If $G$ is edge-weighted, then there exists a perfect matching with weight at most $\frac{1}{2r+1} W$, where $W = \sum_{e \in E(G)} w_e$ and $w_e$ is the weight on an edge $e$. For cubic graphs, the bound reduces to $\frac{1}{3} W$. 
Proof. By Edmonds’ Lemma, $|\mathcal{M}| \frac{n}{2} = \frac{(2r+1)n}{2} p$, which implies that $|\mathcal{M}| = p(2r + 1)$. 
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Since $\sum w_{M_i} = p \sum_{e \in E(G)} w_e = pW$ where $w_{M_i}$ is the total weight of all edges in $M_i$, the pigeonhole principle implies that a matching $M_j$ with the smallest weight in the family has weight at most $\frac{1}{2r+1} W$. 
Chinese Postman Problem in Cubic graphs

$\mathcal{F}_n$: the family of connected cubic graphs with $n$ vertices.
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**Theorem (O and West 2009+)**

If \( G \) is in \( \mathcal{F}_n \) and \( n \geq 10 \), then \( p(G) \leq \frac{2n-5}{3} \),

with equality if \( G \in \mathcal{H}_1' \).

Proof. If \( G \) is in \( \mathcal{F}_n \) and has no balloons or \( n = 10 \), then \( G \) has a perfect matching and \( p(G) = n/2 \leq \frac{2n-5}{3} \).
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Let \( G_1' \) and \( G_2' \) be the graphs obtained from \( G \) by replacing \( G_2 \) and \( G_1 \), respectively, with \( B_1 \).
Chinese Postman Problem in Cubic Graphs

Every parity subgraph of $G'_i$ contains $e$ and uses at least two edges in $B_1$. Hence, $p(G'_i) = p(G_i) + 3$ and $p(G) = p(G'_1) + p(G'_2) - 5$. 
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Letting $n_i = |V(G'_i)|$, we have $n = n_1 + n_2 - 10$.

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**Motivation and Questions**

**Previous Results using Balloons**

**Smallest Matching Number and Edge-Connectivity**

**Chinese Postman Problem**
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Letting $n_i = |V(G'_i)|$, we have $n = n_1 + n_2 - 10$.

By applying the induction hypothesis to both $G'_1$ and $G'_2$,

$$p(G) = p(G'_1) + p(G'_2) - 5 \leq \frac{2n_1 - 5}{3} + \frac{2n_2 - 5}{3} - 5 = \frac{2n - 5}{3}$$
Chinese Postman Problem in Cubic Graphs

Last case: every cut-edge is incident to a copy of $B_1$. 
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The resulting weighted graph $G''$ has a perfect matching with at most $1/3$ of its total weight ($= \frac{m-8b}{3}$).
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Now, $p(G) \leq p(G') + 3b \leq \frac{m-8b}{3} + 3b = \frac{3n-16b}{6} + 3b \leq \frac{2n-5}{3}$. 