Balloons, Chinese Postman Problem and Cycle Double Cover in Cubic Graphs

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Abstract

We previously determined the minimum size of a maximum matching in a connected 
\((2r + 1)\)-regular graph with \(n\) vertices; the extremal graphs have cut-edges. In this 
paper, we prove a lower bound for the minimum size of a maximum matching in a 
t-edge-connected \(r\)-regular graph with \(n\) vertices, for \(t \geq 2\) and \(r \geq 4\). The bound 
is sharp infinitely often and improves a recent result of Henning and Yeo. We also 
study the Chinese Postman Problem, which is the problem of find a shortest closed 
walk traversing all the edges. In a \((2r + 1)\)-regular graph, the problem is equivalent 
to finding a smallest spanning subgraph in which all vertices have odd degree. We 
establish an upper bound for the solution in terms of the edge-connectivity, the vertex 
degree, and the number of vertices. The bound is sharp infinitely often.

1 Introduction

A long time ago, Petersen proved that if a cubic graph has no cut-edges, then it has a 
perfect matching. It is naturally to ask that if there are cut-edges in a cubic graph, then 
what happens in it? If there are cubic graphs with many cut-edges which has no perfect 
matching, then how small can the matching number be in a connected cubic graph with 
\(n\) vertices?, and is there a relationship between the number of cut-edges and the matching 
number in a connected cubic graph? If there is a relationship between them, then how 
many cut-edges can a connected cubic graph have?, and does the relationship give the best

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upper bound for the smallest matching number in a family of connected cubic graph? If the number of cut-edges does not make it best possible, then is there another notion to get the best one? The answers to all above the questions are appeared in the paper [10], and even more generally, for a connected \((2r + 1)\)-regular graph.

In this paper, we naturally study the best lower bound for the matching number of a connected \((2r + 1)\)-regular graph considering edge-connectivity. In the second section, we give the best upper bound, and in the following section, the examples for the best upper bound will be constructed. In the section 4, we introduce a family which appeared in the paper [10], and study the parity number of a graph in the family. The reason why we introduce the family again in this paper is to give the best upper bound for the Chinese Postman Problem in a cubic graph. The Problem was discovered in early 1960's by the Chinese mathematician, Kwan Mei-Ko. Roughly speaking, a Chinese Postman wishes to travel along every road in a city in order to deliver letters, with the least possible distance. More mathematically explaining, a Postman tour in a connected graph \(G\) is an arbitrary closed walk containing all the edges of \(G\). The Chinese Postman Problem is to find a shortest Postman tour in \(G\). Let \(e_c(G)\) be the number of edges in it. A parity subgraph in a graph \(G\) is a spanning subgraph \(H\) of \(G\) such that \(d_G(v) \equiv d_H(v) \mod 2\) for every vertex \(v\) in \(G\). Let \(p(G)\), the parity number of \(G\), be the minimum number of edges in a parity subgraph of \(G\). Since in the Chinese Postman Problem, we are interested in finding the smallest parity subgraph in it, we have the equality that \(e_c(G) = |E(G)| + p(G)\). In view of many applications of the Chinese Postman problem, it is natural to ask for the value of \(e_c(G)\), or equivalently, the value of \(p(G)\). In this paper, we give the best upper bound of the parity number of a connected cubic graph, which gives the best upper bound of \(e_c(G)\) in a connected cubic graph \(G\).

2 Matching

In the following theorem 2.1, we also have the lower bound for the smallest matching number of a connected \((2r + 1)\)-regular graph, i.e. when \(t = 0\). However, the bound is sharp only for finite cases, especially when \(G\) is a connected \((2r + 1)\)-regular graph, and \(K_{1,2r+1}\) is the graph obtained from \(G\) by deleting all the balloons. Thus, the following theorem 2.1 does not implies our previous result in the paper [10]. Nevertheless, the proof of the following theorem 2.1 gives the best lower bound considering the other edge-connectivity.

**Theorem 2.1.** If \(G\) is a \((2t + 1)\)-edge-connected \((2r + 1)\)-regular graph with \(n\) vertices, then \(\alpha'(G) \geq \frac{n}{2} - \left(\frac{r-t}{2^{(t+1)^2+t}}\right)\frac{n}{2}\), and this is sharp infinitely often for \(t \geq 1\).

**Proof.** Let \(S\) be a set with maximum deficiency. Thus, \(\alpha'(G) = \frac{1}{2}(n - \text{def}(S))\), where \(\text{def}(S) = o(G - S) - |S|\). Let \(c_i\) count the odd components of \(G - S\) having \(i\) edges to \(S\).
Let \( c = c_{2i+1} + \ldots + c_{2i-1} \), and let \( c' = o(G - S) - c \). Note that for \((2t+1) \leq i \leq 2r-1\), each odd component of \( G - S \) having \( i \) edges to \( S \) has at least \((2r+3)\) vertices. (Otherwise, each vertex of \( G \) has a neighbor outside.) Counting the edges joining \( S \) to odd components of \( G - S \) yields \((2r+1)|S| \geq (2r+1)c' + (2t+1)c \) and hence \(|S| \geq c' + \left( \frac{2t+1}{2r+1} \right) c \geq \left( \frac{2t+1}{2r+1} \right) c \). Therefore, \( n \geq |S| + c(2r+3) \geq \left( \frac{2t+1}{2r+1} \right) c + c(2r+3) \), which implies that \( c \leq \left( \frac{2r+1}{4r^2 + 4r + 4 + 2t} \right)n \).

Now, we compute

\[
\text{def}(S) = (c + c') - |S| \leq c - \frac{2t+1}{2r+1} c = \frac{2(r-t)}{2r+1} c
\]

\[
\leq \frac{2(r-t)}{2r+1} \left( \frac{2r+1}{4r^2 + 4r + 4 + 2t} \right)n = \frac{(r-t)n}{2(r+1)^2 + t}.
\]

\[\square\]

**Corollary 2.2.** Let \( G \) be a \((2t+1)\)-regular graph with \( n \) vertices for \( 1 \leq t \leq r \) and \( r \geq 2 \). Then \( \alpha'(G) \geq \left( \frac{2r^2 + 2t}{2(2r+1)^2 + t} \right)n \).

**Corollary 2.3.** Let \( G \) be a \((2t)\)-regular graph with \( n \) vertices for \( 1 \leq t \leq r \) and \( r \geq 2 \). Then \( \alpha'(G) \geq \left( \frac{2r^2 + 2t}{2(2r+1)^2 + t} \right)n \).

**Corollary 2.4.** Let \( G \) be a \((2t-1)\)-regular graph with \( n \) vertices for \( 1 \leq t \leq r \) and \( r \geq 2 \). Then \( \alpha'(G) \geq \left( \frac{2r^2 + 2t}{2(2r+1)^2 + t} \right)n \).

By the above results, and O and West [10] we have another proof that \((r-1)\)-edge-connected \( r \)-regular graph has a perfect matching for \( r \geq 4 \) and \( r \geq 3 \).

## 3 Chinese Postman Problem

In this section, we introduce the examples achieving the bound of the theorem ??.

Let \( B_{r,t} \) be the resulting graph obtained from the graph \( B_r \) by deleting a matching with \( t \) edges in \( B_r \).

**Proposition 3.1.** For \( 0 \leq t \leq r \), the edge connectivity of the graph \( B_{r,t} \) is \( 2r \).

**Proof.** Assume to the contrary that the edge-connectivity of the graph \( B_{r,t} \) is less than \( 2r \). Then there exists a subset \( S \) in \( V(G) \) such that \(|\{S, \overline{S}\}| < 2r \). Notice that for \( 0 \leq t \leq r \), all the vertices in the graph \( B_{r,t} \) has degree at least \( 2r \), \( |V(B_{r,t})| = 2r + 3 \), and \( |S| > 1 \). We can easily notice that \(|\{S, \overline{S}\}| = \sum_{v \in S} \deg(v) - 2|E(G[S])| \). Since \( \deg v \geq 2r \), \( |S|(|S| - 1) \geq 2|E(G[S])| \), and \( |S| > 1 \), \( 2r > |\{S, \overline{S}\}| = \sum_{v \in S} \deg(v) - 2|E(G[S])| \geq 2r|S| - |S|(|S| - 1) \). Thus, \(|S| > 2r \) and similarly we have \(|\overline{S}| > 2r \) which is impossible. Finally, \( \kappa'(B_{r,t}) = 2r \) since \( \delta(B_{r,t}) = 2r \). \[\square\]
Let $A = \{a_{11}, a_{12}, \ldots, a_{1(2t+1)}, \ldots, a_{k1}, \ldots, a_{k(2t+1)}\}$, and let $B = \{b_{11}, b_{12}, \ldots, b_{1(2r+1)}, \ldots, b_{k1}, \ldots, b_{k(2r+1)}\}$.

Add an edge from $a_{ij}$ to $b_{il}$ if $j \neq l$ and from $b_{il}$ to $a_{(i+1)l}$ (mod $k$). Let $H_{r,t,k}'$ be the resulting graph, and notice that each vertex in $A$ and $B$ has degree $2r+1$ and $2t+1$, respectively. Now, let the graph $G_{r,t,k}$ be the resulting graph obtained from $H_{r,t,k}'$ by expanding each vertex in $B$ to $B_{r,t}$, is $(2t+1)$-edge connected.

**Proposition 3.2.** The graph $G_{r,t,k}$ is $(2t+1)$-edge connected.

**Proof.** Let $S$ be a vertex subset in $G_{r,t}$ such that $|S, \bar{S}| < 2t+1$.

By expanding each vertex in $B$ to $B_{r,t}$, we have an infinite family of $(2t+1)$-edge-connected $(2r+1)$-regular graphs.

## 4 Chinese Postman Problem

### 5 Examples

In the paper [10], we constructed a certain family of graphs holding the lower bound of a matching number in the family of connected $(2r+1)$ regular graphs with $n$ vertices. In this section, we introduce the family. Interestingly, the family is useful to study the Chinese Postman Problem, and in fact, the bound for the problem in the family of cubic graphs with $n$ vertices hold in the family.

**Example 5.1.** Let $B_r$ be the graph obtained from the complete graph $K_{2r+3}$ by deleting a matching of size $r + 1$ and one more edge incident to the remaining vertex. This is the smallest graph in which one vertex has degree $2r$ and the others have degree $(2r+1)$. Thus $B_r$ is the smallest possible balloon in a $(2r+1)$-regular graph. Note that deleting the vertex of degree $2r$ (the neck) from $B_r$ leaves a subgraph having a perfect matching.

Let $T_r'$ be the family of trees such that every non-leaf vertex has degree $2r + 1$. Let $H_r'$ be the family of $(2r+1)$-regular graphs obtained from trees in $T_r'$ by identifying each leaf of such a tree with the neck in a copy of $B_r$.

We [10] studied on the family and the following proposition is a part of the study.

**Proposition 5.2.** [10] Let $p_r = 2r^2 + 2r - 1$. For any $n$-vertex graph $G$ in $H_r'$,

$$b(G) = \frac{(2r-1)n+2}{2p_r}, \quad c(G) = \frac{r(n-2)-2}{p_r} - 1.$$ 

**Lemma 5.3.** Let $G$ be in $H_r'$, and $T$ be the tree obtained by shrinking each $B_r$ to a single vertex. Then every parity subgraph of $G$ has $T$ as its subgraph.
Proof. Let $B_r$ be a subgraph in $G$ which is incident to the cut-edge $e$. If $P$ is a parity subgraph of $G$, then $P$ has to use the edge $e$. To keep parity of each vertex in $P$, we have to use all the edges in $T$.

**Proposition 5.4.** Let $G$ be in $H'_r$. Then

$$p(G) = \frac{(2r^2 + 3r - 1)n - 2(r + 1)}{4r^2 + 4r - 2} - 1$$

Proof. Let $T'_r$ be the tree obtained by deleting all the balloon in $G$. By Lemma 5.3, we have to use all the edges in $T'_r$ to have a parity subgraph of $G$. Notice that since we used an edge $e$ which is incident to $B_r$ in $G$, a smallest parity subgraph in $B_r$ has $r + 1$ edges. Hence,

$$p(G) = c(G) + (r + 1)b(G) = \frac{r(n - 2) - 2}{p_r} - 1 + (r + 1)\frac{(2r - 1)n + 2}{2p_r}$$

$$= \frac{2r(n - 2) - 4 + (r + 1)(2r - 1)n + 2}{2p_r} - 1 = \frac{(2r^2 + 3r - 1)n - 2(r + 1)}{4r^2 + 4r - 2} - 1$$

6 Balloons and Chinese Postman Problem

A Postman tour in a connected graph $G$ is an arbitrary closed walk containing all the edges of $G$. The Chinese Postman Problem is to find a shortest Postman tour in $G$. Let $e_c(G)$ be the number of edges in it. A parity subgraph in a graph $G$ is a spanning subgraph $H$ of $G$ such that $d_G(v) \equiv d_H(v) \mod 2$ for every vertex $v$ in $G$. Let $p(G)$ be the minimum number of edges in a parity subgraph of $G$. It is trivial that $e_c(G) = |E(G)| + p(G)$. In view of many applications of the Chinese Postman problem, it is natural to ask what the value of $e_c(G)$ is, or equivalently, what the value of $p(G)$ is. Let $F_n$ be the family of all connected cubic graphs with $n$ vertices. In this paper, we give the upper bounds for the value $p(G)$ for $G \in F_n$, and the bound is infinitely many sharp for graphs in $H'_1$.

**Definition 6.1.** Given $S, T \subseteq V(G)$, we write $[S, T]$ for the set of edges having one end vertex in $S$, and the other in $T$.

**Definition 6.2.** An $r$-graph $G$ with $n$ vertices is an $r$-regular graph such that for each vertex subset $S \subseteq V(G)$ with $|S| \equiv 1 \mod 2$ and $0 < |S| < n$, $|[S, V(G) - S]| \geq r$.

Notice that if $G$ is a 2-edge-connected multi cubic graph, then $G$ is 3-graph.

More generally, if $G$ is an $(r-1)$-edge-connected multi $r$-regular graph, then $G$ is $r$-graph.
Lemma 6.3. [5] Let $G$ be an $r$-graph. Then there is an integer $p$ and a family $\mathcal{M}$ of perfect matchings such that each edge of $G$ is contained in precisely $p$ members of $\mathcal{M}$. (Note that it is not necessary that the members of $\mathcal{M}$ must be distinct.

Lemma 6.4. Let $G$ be a $2r$-edge-connected multi-$(2r+1)$ regular graph. If $G$ is edge-weighted, then there exists a perfect matching with at most $\frac{1}{2r+1}w_t$ where $w_t = \sum_{e \in E(G)} w_e$ and $w_e$ is the weight on an edge $e$.

Proof. By Lemma 6.3, there are $p$ perfect matchings in $G$ and notice that $p = (2r+1)q$ for a positive integer $q$. Let $\mathcal{M} = \{M_1, M_2, ..., M_p\}$ be the family of such $p$ perfect matchings in the lemma 6.3. Let $w_E = \sum_{e \in E} w_e$ where $E$ is a set of edges, and let $M = \min_{i \in [p]} w_{M_i}$. Without loss of generality, we may assume that $M_1$ has the total sum $M$. Then we have $pM \leq \sum_{i \in [p]} w_{M_i} = q \sum_{e \in E(G)} w_e = qw_T$. Therefore, $M \leq \frac{1}{2r+1}w_T$ which implies that there exists such a perfect matching.

Corollary 6.5. Let $G$ be a 2-edge-connected multi-cubic graph. If $G$ is edge-weighted, then there exists a perfect matching with at most $\frac{1}{3}w_t$ where $w_t = \sum_{e \in E(G)} w_e$ and $w_e$ is the weight on an edge $e$.

Theorem 6.6. For $G$ in $\mathcal{F}_n$, there exists a parity subgraph $p(G)$ such that $|E(p(G))| \leq \frac{2n-5}{3}$, with equality if $G \in \mathcal{H}_1$.

Proof. Notice that if $G \in \mathcal{F}_n$ and has no balloons, then $G$ has a perfect matching which implies that $p(G) = n/2$. So we may assume that $G$ has a balloon. We use induction on $n$. If $n \leq 10$, then $G$ has a perfect matching which implies that $p(G) = n/2$. Hence all the claim hold for the basis. Now, suppose that $n > 10$ and $e$ is a cut-edge in $G$. Let $G_1$ and $G_2$ be the components of $G - e$. Let $G'_1$ and $G'_2$ be the graphs obtained from $G$ by replacing $G_2$ and $G_1$, respectively, with $B_1$. First, notice that $p(G'_i) = p(G_i) + 3$ since any parity subgraph of $G'_i$ has to have the edge $e$, and if we used the edge, then the parity subgraph has to use at least two edges in $B_1$. Hence, $p(G) = p(G_1) + p(G_2) - 5$. If neither $G_1$ nor $G_2$ equals $B_1$, then $G'_1$ and $G'_2$ have fewer vertices than $G$, and letting $n_i = |V(G'_i)| = n_i$, we have $n = n_1 + n_2 - 10$. By applying the induction hypothesis to both,

$$p(G) = p(G'_1) + p(G'_2) - 5 \leq \frac{2n_1 - 5}{3} + \frac{2n_2 - 5}{3} - 5 = \frac{2(n_1 + n_2 - 10) + 10}{3} - 5 = \frac{2n - 5}{3}$$

In the remaining case, every cut-edge in $G$ is incident to a copy of $B_1$ which implies that $c(G) = b(G)$. Now, suppose that each edge in $G$ has weight 1. Let $G'$ be the resulting graph obtained by deleting all the balloons. blablabla and $G''$ has a perfect matching with at most $1/3$ weight $(\frac{m-8b}{3})$ by Lemma 6.4. Then

$$p(G) \leq p(G') + 3b \leq \frac{m-8b}{3} + 3b = \frac{3n - 16b}{6} + 3b = n/2 + b/3 \leq \frac{n}{2} + \left(\frac{n+2}{6}\right)\frac{1}{3} \leq \frac{2n - 5}{3}$$
Corollary 6.7. If $G \in \mathcal{F}_n$, then $e_P(G) \leq \frac{13n-10}{6}$. Equality holds infinitely often.

Definition 6.8. We call a subgraph $B_t$ of $G$ t-balloon if $B_t$ is $(t + 1)$-edge-connected and is incident to exactly $(t + 1)$ edges in $G$.

Lemma 6.9. Let $G$ be a connected $(2r+1)$-regular graph (allowed multiple edges, or loops) with at least $4r + 6$ vertices. Then there exists a $B_t$ of $G$ for some odd $t$ with $|V(B_t)| \geq 2$ and $|V(G - B_t)| \geq 2$.

Theorem 6.10. Let $G$ be a connected $(2r+1)$-regular graph, then there exists a parity subgraph $P(G)$ such that $|E(P(G))| \leq \frac{(2r^2 + 3r - 1)n - 2(r+1)}{4r^2 + 4r - 2} - 1$

Proof. Assume that $G$ has the edge-connectivity $t$. Let $B$ be a smallest $t$-balloon for smallest odd $t < 2r + 1$. Let $G_1$ and $G_2$ be the resulting graphs obtained from contracting $B$ to a single vertex $x$ and adding some loops making $G_1$ a connected $(2r+1)$-regular graph, and contracting $G - B$ to a single vertex $y$ adding some loops making $G_2$ a connected $(2r+1)$-regular graph. Notice that $n = n_1 + n_2 - 2$. Claim that $|E(P(G))| = |E(P(G_1))| + |E(P(G_1))| - 1$ if the claim holds, then by induction step on $n$, we have the desired result.

$$|E(P(G))| = |E(P(G_1))| + |E(P(G_1))| - 1$$

$$\leq \frac{(2r^2 + 3r - 1)n_1 - 2(r+1)}{4r^2 + 4r - 2} - 1 + \frac{(2r^2 + 3r - 1)n_2 - 2(r+1)}{4r^2 + 4r - 2} - 1 - 1$$

$$\leq \frac{(2r^2 + 3r - 1)n - 2(r+1)}{4r^2 + 4r - 2} - 1 + \frac{2(2r^2 + 3r - 1) - 2(r+1)}{4r^2 + 4r - 2} - 2$$

$$\leq \frac{(2r^2 + 3r - 1)n - 2(r+1)}{4r^2 + 4r - 2} - 1$$

Theorem 6.11. If $G$ is a $(2t+1)$-edge-connected $(2r+1)$-regular graph for $t \geq 1$, then there exists a parity subgraph $P(G)$ such that $|E(P(G))| \leq \frac{(2r+1)^2 + r}{4(r+1)^2 + 2r}$

References


