Lesson A3: Euler-Lagrange **Unit A: Geodesics**

Objective. At the end of this lesson, students will be able to:

- Derive Euler-Lagrange equations
- Derive and check the geodesic equations.

Assignments:

- Read Jost pages 18-19. Wikipedia article: "Euler-Lagrange"

Materials and Equipment needed:

- none

Introduction.

Las time we defined the distance between two points on a Reimannnian as the infimimum of the lengths of the paths between then. We showed that this defines a metric on on the manifold which is consistent with the manifold topology.

Today we will derive the geodesic equations and show that the shortest path between any two points is a geodesic.

Outline.

Euler Lagrange equations: *30 min lecture.*

Wikipedia page on Lagrangian mechanics:

Fix a **Lagrangian**: a smooth map $L\colon \mathbb{R} \times TM \to \mathbb{R}$.
Fix $a < b \in \mathbb{R}$ and $p, q \in M$ and define

$$\mathcal{P} := \{\text{piecewise smooth } \gamma\colon [a,b] \to M \mid \gamma(a) = p, \gamma(b) = q\}.$$

Define an **action functional** $A\colon \mathcal{P} \to \mathbb{R}$ by

$$A(\gamma) = \int_a^b L(t, \gamma'(t))dt.$$

Given a coordinate chart on M:

- we obtain coordinates $x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n$ on TM;
- let $\dot{\gamma}^i = \frac{d}{dt}\gamma^i$.

We say $\gamma \in \mathcal{P}$ is **stationary** if, for all coordinate charts

$$\frac{\partial L}{\partial x^i}(t, \gamma(t), \dot{\gamma}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}(t, \gamma(t), \dot{\gamma}(t)) = 0 \quad \forall i.$$

Lemma. *If γ is an extremal point of A, then γ is stationary.*

Proof. If γ is extremal, then $\gamma|_{[a',b']}$ is extremal for all $a \le a' < b' \le b$.
Therefore, by restricting to a coordinate chart, WLOG γ is smooth and $M \subseteq \mathbb{R}^n$.
Fix $\eta\colon [a,b] \to \mathbb{R}^n$ with $\eta(a) = \eta(b) = 0$.
For sufficiently small $s \in \mathbb{R}$, $\gamma + s\eta$ is a smooth curve in M.
Define $\hat{A}(s) \in \mathbb{R}$ by

$$\hat{A}(s) = A(\gamma + s\eta).$$

Since γ is an extremal point of A, 0 is an extremal point of $\hat{A}$.

$$\implies 0 = \frac{d\hat{A}}{ds}\Big|_{s=0} = \frac{d}{ds}\int_a^b L\big(t, \gamma(t) + s\eta(t), \dot{\gamma}(t) + s\dot{\eta}(t)\big)dt\Big|_{s=0}$$
$$= \int_a^b \frac{\partial}{\partial s}\big(L(t, \gamma(t) + s\eta(t), \dot{\gamma}(t) + s\dot{\eta}(t)\big)dt\Big|_{s=0}$$
$$= \int_a^b \frac{\partial L}{\partial x^i}(t, \gamma(t), \dot{\gamma}(t))\,\eta^i(t) + \frac{\partial L}{\dot{x}^i}(t, \gamma(t), \dot{\gamma}(t))\,\dot{\eta}^i(t)dt$$

Since $\eta(a) = \eta(b) = 0$, by integration by parts yields

$$\int_a^b \frac{\partial L}{\dot{x}^i}(t, \gamma(t), \dot{\gamma}(t))\,\dot{\eta}^i(t)dt = \frac{\partial L}{\dot{x}^i}(t, \gamma(t), \dot{\gamma}(t))\,\eta^i(t)\Big|_a^b - \int_a^b \frac{d}{dt}\left(\frac{\partial L}{\dot{x}^i}(t, \gamma(t), \dot{\gamma}(t))\right)\eta^i(t)dt$$
$$= -\int_a^b \frac{d}{dt}\left(\frac{\partial L}{\dot{x}^i}(t, \gamma(t), \dot{\gamma}(t))\right)\eta^i(t)dt$$

Therefore,

$$\int_a^b \left(\frac{\partial L}{\partial x^i}(t, \gamma(t), \dot{\gamma}(t)) - \frac{d}{dt}\frac{\partial L}{\dot{x}^i}(t, \gamma(t), \dot{\gamma}(t))\right)\eta^i(t)dt = 0$$

for all i and all $\eta\colon [a, b] \to \mathbb{R}^n$ with $\eta(a) = \eta(b) = 0$.

Hence, by the Fundamental Lemma of the calculus of variations, γ is stationary.

□

Energy: *20 min lecture.*

Let (M, g) be a Reimannian manifold.

Let $||v||^2 = g(v, v)$.

Consider a smooth curve $\gamma\colon [a, b] \to M$:

The **length** of γ is

$$L(\gamma) = \int_a^b ||\gamma'(t)||dt.$$

The **energy** of γ is

$$E(\gamma) = \frac{1}{2}\int_a^b ||\gamma'(t)||^2 dt.$$

In both cases, sum over pieces for piecewise smooth curves.

Lemma.

$$L(\gamma)^2 \leq 2(b - a)E(\gamma)$$

with equality iff $||\gamma' t(t)||$ *is constant.*

If (g_{ij}) is the $n \times n$ matrix representing g:

- Let $(g^{ij}) = (g_{ij})^{-1}$, i.e. $g^{i\ell}g_{\ell j} = \delta_{ij}$.
- Let $g_{ij,k} := \frac{\partial g_{ij}}{\partial x^k}$.

Lemma. *The Euler-Lagrange equations for the energy E are*

$$(\star) \qquad \ddot{\gamma}^i(t) + \Gamma^i_{jk}(\gamma(t))\dot{\gamma}^j(t)\dot{\gamma}^k(t) = 0 \quad i \in \{1, \ldots, d\}.$$

where $\Gamma^i_{jk} = \frac{1}{2}g^{i\ell}(g_{j\ell,k} + g_{k\ell,j} - g_{jk,\ell})$ *are the Christoffel symbols.*

Proof. The Lagrangian for $2E$ is

$$L(t,x,\dot{x}) = g_{jk}(x)\dot{x}^j\dot{x}^k$$

$$\implies \frac{\partial L}{\partial x^i} = g_{jk,i}(x)\dot{x}^j\dot{x}^k \quad \text{and} \quad \frac{\partial L}{\partial \dot{x}^i} = g_{ik}(x)\dot{x}^k + g_{ji}(x)\dot{x}^j$$

Therefore, the Euler-Lagrange equation for E is

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}(t,\gamma(t),\dot{\gamma}(t)) - \frac{\partial L}{\partial x^i}(t,\gamma(t),\dot{\gamma}(t)) \\ &= g_{ik,j}(\gamma)\dot{\gamma}^j\dot{\gamma}^k + g_{ik}(\gamma)\ddot{\gamma}^k + g_{ji,k}(\gamma)\dot{\gamma}^j\dot{\gamma}^k + g_{ji}(\gamma)\ddot{\gamma}^j - g_{jk,i}(\gamma)\dot{\gamma}^j\dot{\gamma}^k \\ &= 2g_{ik}(\gamma)\ddot{\gamma}^k + \big(g_{ik,j}(\gamma) + g_{ji,k}(\gamma) - g_{jk,i}(\gamma)\big)\dot{\gamma}^j\dot{\gamma}^k \quad \forall i. \end{aligned}$$

Replacing i by ℓ and multiplying by $\frac{1}{2}(g^{i\ell})$

$$\begin{aligned} 0 &= g^{i\ell}(\gamma)g_{\ell k}(\gamma)\ddot{\gamma}^k + \frac{1}{2}g^{i\ell}(\gamma)\big(g_{\ell k,j}(\gamma) + g_{j\ell,k}(\gamma) - g_{jk,\ell}(\gamma)\big)\dot{\gamma}^j\dot{\gamma}^k \\ &= \ddot{\gamma}^i(t) + \frac{1}{2}g^{i\ell}\big(g_{k\ell,j}(\gamma(t)) + g_{j\ell,k}(\gamma(t)) - g_{jk,\ell}(\gamma(t))\big)\dot{\gamma}^j\dot{\gamma}^k. \end{aligned}$$

□

We call solutions to $(\star)$ **geodesics**

Proposition. *If $\gamma\colon [a,b] \to M$ is the shortest path from p to q, then γ is a geodesic.*

Proof. The length functional is invariant under reparameterization.
Moreover, every regular curve may be reparametrized by arc length.
Therefore, we may assume WLOG that γ is parameterized by arc length.
Hence if $\tilde{\gamma}\colon [a,b] \to M$ is any curve from p to q, then

$$E(\gamma) = \frac{1}{2}(b-a)L(\gamma)^2 \le \frac{1}{2}(b-1)L(\tilde{\gamma})^2 \le E(\tilde{\gamma}).$$

Therefore γ is an extremal point of E.
Hence, it satisifies the Euler-Lagrange equation of E. □

Lemma. *Every geodesic is parameterised proportionatelly to arc length.*

Proof. If γ is a geodesic,

$$\begin{aligned} \frac{d}{dt}\langle\dot{\gamma},\dot{\gamma}\rangle &= \frac{d}{dt}\big(g_{ij}(\gamma)\dot{\gamma}^i\dot{\gamma}^j\big) = g_{ij}\ddot{\gamma}^i\dot{\gamma}^j + g_{ij}\dot{\gamma}^i\ddot{\gamma}^j + g_{ij,k}\dot{\gamma}^i\dot{\gamma}^j\dot{\gamma}^k \\ &= -g_{ij}\big(g^{im}\big(g_{km,\ell} + g_{\ell m,k} - g_{k\ell,m}\big)\dot{\gamma}^k\dot{\gamma}^\ell\big)\dot{\gamma}^j + g_{ij,k}\dot{\gamma}^i\gamma^j\dot{\gamma}^k \\ &= -\big(g_{kj,\ell} + g_{\ell j,k} - g_{k\ell,j}\big)\dot{\gamma}^j\dot{\gamma}^k\dot{\gamma}^\ell + g_{ij,k}\dot{\gamma}^i\gamma^j\dot{\gamma}^k = 0. \end{aligned}$$

□

Theorem. *Fix $p \in M$ and $v \in T_pM$. For sufficiently small $\epsilon > 0$, there is a unique geodesic $c_v\colon [0,\epsilon] \to M$ with $c_v(0) = 0$ and $\dot{c}_v(0) = v$. Additionally, v depends smoothly on p and v.*

Proof. $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is a solution to the differential equations $\ddot{\gamma}^i = -\Gamma^i_{jk}(\gamma)\dot{\gamma}^j\dot{\gamma}^k$ if and only if $(\gamma,\eta)\colon \mathbb{R} \to \mathbb{R}^{2n}$ is a solution to the linear differential equations $\dot{\gamma}^i = \eta_i$, $\dot{\eta}^i = -\Gamma^i_{jk}(\gamma)\eta^j\eta^k$. Hence, this is an immediate consequence of the theorem of existence and uniqueness of differential equations. □

Given $p \in M$, let

$$V_p = \{v \in T_pM \colon c_v \text{ is defined on } [0,1]\}.$$

Define the **exponential map** $\exp_p \colon V_p \to M$ by $v \mapsto c_v(1)$.

Note: Given an open set $0 \in U \subseteq \mathbb{R}^n$ with the Euclidean metric, if $U \neq \mathbb{R}^n$ then

$$V_0 = \{x \in \mathbb{R}^n \mid [0,x] \subseteq U\} \neq T_0U.$$

Theorem. $\exp_p$ *is a diffeomorphism from a neighborhood of* $0 \in T_pM$ *onto a neighborhood* U *of* $p \in M$.

Proof. If γ is a geodesic, then so is $t \mapsto \gamma(\lambda t)$ for $\lambda \in \mathbb{R}$.

Hence, $c_{\lambda v}(t) = c_v(\lambda t)$, and so $c_{\lambda v}$ is defined on $[0, \frac{\epsilon}{\lambda}]$.

Since $\{v \in T_pM \colon ||v|| = 1\}$ is compact and c_v depends smoothly on v, there exists $\epsilon > 0$ so that c_v is deifned on $[0, \epsilon]$ for all $v \in T_pM$ with $||v|| = 1$. Therefore, c_w is defined on $[0,1]$ for all $w \in T_pM$ with $||w|| \leq \epsilon$. Hence, V_p contains a neighborhood of $0 \in T_pM$.

If we identify T_0T_pM with T_pM, then

$$d\exp_p(0)(v) = \frac{d}{dt}c_{tv}(1)\Big|_{t=0} = \frac{d}{dt}c_v(t)\Big|_{t=0} = \dot{c}_v(0) = v.$$

$$\implies d\exp_p(0) = \mathrm{Id}\Big|_{T_pM}$$

Therefore, $d\exp_p(0)$ has maximal rank, so the theorem follows from the implicit function theorem. □

HW: Describe all geodesics with unit speed through the south pole of $S^n \subseteq \mathbb{R}^{n+1}$. Verify the geodesic equations using stereographic projection from the north pole.