Homework #4 Solutions

(1) **Theorem.** On $\mathbb{E}^2$, $\mathbb{H}^2$, or $\mathbb{S}^2$, $\angle BAC \simeq \angle B'A'C'$ if and only if $m \angle BAC = m \angle B'A'C'$.

**Proof.** Assume first that $\angle BAC \simeq \angle B'A'C'$. Then by definition there exists an isometry $f$ so that $f(\angle BAC) = \angle B'A'C'$. Since isometries preserve angle measure, this implies that $m \angle BAC = m \angle B'A'C'$.

So assume instead that $m \angle BAC = m \angle B'A'C'$. By the Ray Congruence Theorem, there exists an isometry $f$ which takes $\overrightarrow{A'B'}$ to $\overrightarrow{AB}$. If $f(\overrightarrow{A'C'})$ and $\overrightarrow{AC}$ lie on opposite sides of $\overrightarrow{A'B'}$, let $g = R \circ f$, where $R$ is the reflection over $\overrightarrow{AB}$. Otherwise, let $g = f$. Then $g$ takes $\overrightarrow{AB}$ to $\overrightarrow{A'B'}$; moreover $g(\overrightarrow{A'C'})$ and $\overrightarrow{AC}$ lie on the same side of $\overrightarrow{AB}$. Finally, since $m \angle BAC = m \angle B'A'C'$ and isometries preserve measure, $m \angle B'A'C' = m \angle BAC$.

Hence, by the protractor axiom, $g(\overrightarrow{A'C'}) = \overrightarrow{AC}$. $\square$

(2) (1) The fewest restriction is a) and b).

(2) **Theorem.** Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles on the sphere such that each edge has length less than half the circumference. If $m \overrightarrow{AB} = m \overrightarrow{A'B'}$, $m \angle BAC = m \angle B'A'C'$, and $m \overrightarrow{AC} = m \overrightarrow{A'C'}$, then $\triangle ABC \simeq \triangle A'B'C'$.

**Proof.** Since $m \angle BAC = m \angle B'A'C'$, ACT implies that there exists an an isometry $f$ taking $\angle B'A'C'$ to $\angle BAC$. More precisely, $f(\overrightarrow{A'B'}) = \overrightarrow{AB}$, and $f(\overrightarrow{A'C'}) = \overrightarrow{AC}$. Because there is a unique segment along a ray with a given length and isometries preserves length, and since $m \overrightarrow{AB} = m \overrightarrow{A'B'}$ and $m \overrightarrow{AC} = m \overrightarrow{A'C'}$, this implies that we must have $f(\overrightarrow{A'B'}) = \overrightarrow{AB}$ and $f(\overrightarrow{A'C'}) = \overrightarrow{AC}$. Finally, $f(\overrightarrow{BC'})$ and $\overrightarrow{BC}$ are both segments connecting $B$ and $C$. Moreover, given any two points $B$ and $C$ on the sphere, there is at most one segment joining them with length less than $S$. Therefore, since $m \overrightarrow{BC} < S$ and $m \overrightarrow{BC'} = m f(\overrightarrow{BC'}) < S$ by assumption, we conclude that $f(\overrightarrow{BC'}) = \overrightarrow{BC}$. $\square$

(a) Pick three points $A$, $B$, and $C$ that are not collinear. Let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ be the shortest segments joining their endpoints. Then by alternatively taking the short or long segment $\overrightarrow{BC}$, we can get two triangles that satisfy the assumptions of SAS but are not congruent. In this case, the problem is that, even though there is a unique line joining $B$ and $C$, there are two segments joining them because the ruler axiom is false. Now let $B$ and $C$ be antipodal points, but again let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ be the shortest segments joining their endpoints. Then, since the incidence axiom fails, there is more than one (in fact, infinitely many) joining $B$ and $C$, and hence more than one segment joining them. By picking two different segments, we can get two triangles that satisfy the assumptions of SAS but are not congruent.
(3) Consider a $X^\circ$ cone.
(a) There is at least one straight line between two points exactly if $X \leq 360$.
(b) There is at most one straight line between two points exactly if $X \geq 350$. 