Subsets

Often we deal with several sets at once and study the relationships between them. Here is one relationship that is probably already familiar to you.

2.2.1 DEFINITION
If A and S are sets, we say that S is a subset of A if every element of S is in A (which can be written in implication form as "if x ∈ S, then x ∈ A"). We denote this by \( S \subseteq A \).

The next theorem shows that every nonempty set has at least two subsets.

2.2.2 THEOREM
For all sets \( X, \emptyset \subseteq X \) and \( X \subseteq X \). (Hint: Look at the implication given in the definition of subset. Explicitly write out the implications that you are meant to prove. What can you conclude?)

2.2.3 EXERCISE
How many subsets does \( \emptyset \) have?

Notice that the statement "If \( x \in S \), then \( x \in X \)" is an implication just like those that we studied in the last chapter. Thus to show that \( S \subseteq X \), we must take an \( x \) for which the hypothesis is true (that is, an \( x \) in \( S \)) and show that the conclusion is also true for that \( x \).
That is, we must show \( x \in X \). We do this so often in mathematics that there is a name for the process. It is called an element argument. To get started, try your hand at the following (very simple) element argument.

2.2.4 EXERCISE
Prove that if \( A \subseteq B \) and \( B \subseteq C \), then \( A \subseteq C \). (Notice that your goal is to prove that \( A \subseteq C \). Therefore, according to the process described above, you begin your argument with "Let \( x \in A \)." At the end of your argument, you should be able to say: "Then \( x \in C \)."

2.2.5 DEFINITION
If \( B \) is a subset of \( X \) and \( B \neq X \), then we say that \( B \) is a proper subset of \( X \).

2.2.6 EXERCISE
Give two proper subsets of the set \{1, 3, 5, 7, 9, 11\}.

Because a set is completely characterized by its elements, it is reasonable to say that two sets are the same set if they have the same elements. That is, every element of \( A \) is in \( B \) and every element of \( B \) is in \( A \).

2.2.7 DEFINITION
Suppose that \( A \) and \( B \) are sets. Then \( A = B \) if \( A \subseteq B \) and \( B \subseteq A \).

A word about definitions: Although definitions are written in the same form as theorems, they are fundamentally different. Suppose we have the following definition: "If \( X \) is a clacking waggler and all subsets of \( X \) are marinit, then \( X \) is a supreme clacking waggler." Because we are giving a definition, we are saying exactly what we mean by supreme clacking waggler. We are not just describing some possibility for supreme clacking waggler. The statements

"\( X \) is a supreme clacking waggler;" and
"\( X \) is a clacking waggler and all subsets of \( X \) are marinit"

mean exactly the same thing. Definitions are always equivalences. By convention, the if and only if is understood and never said explicitly.

2.3 Set Operations

The axioms of set theory allow us to "build new sets from old ones." They tell us that any subset of a set that we have at our disposal is also a set. They also allow us to take unions, intersections, and complements of sets.

2.3.1 DEFINITION
Let \( U \) be a set. Let \( S \subseteq U \). Define

\[ S^C = \{ x \in U : x \notin S \}. \]

The set \( S^C \) is called the complement of \( S \) in \( U \). If the set \( U \) is understood, we may just call \( S^C \) "the complement of \( S \)" and denote it by \( S^C \).

(For technical reasons having to do with set-theoretic paradoxes, complements must always be taken relative to a larger set—see Section 2.6. In the absence of the set \( U, S^C \) makes no sense.)

2.3.2 EXERCISE
Consider the intervals \( U = [-5, 5] \) and \( S = [-5, 2] \). Find \( S^C \).

2.3.3 EXERCISE
As in Exercise 2.3.2, let \( S = [-5, 2] \). What is \( S^C \)?