is an important difference between a statement that is false when "Marlene has brown hair" is true, and the negation of "Marlene has brown hair.""

(For a more mathematical, but parallel, example: Ask yourself why \( x = 3 \) is not the negation of \( x = 7 \). What is the negation of \( x = 7 \)?)

7. Suppose that \( (a, b) \) and \( (c, d) \) are two distinct points in \( \mathbb{R}^2 \). Use the processes described in Sections 1.9 and 1.10 to prove that there exists a unique line passing through the two points.

(Remember that the week you do to determine the candidate in the existence part is not part of your proof.)

8. Describe what you would have to do to show that an object is not unique.

9. Your goal will be to prove that "If \( x \) is an odd integer, then \( x^2 \) is an odd integer."

(a) Here are two possible definitions for an odd integer:

* An integer \( z \) is odd if it is not even.
* An integer \( z \) is odd if there exists an integer \( w \) such that \( z = 2w + 1 \).

Which of these two definitions do you think will be more useful to you in the proof? Why?

(b) Prove that the square of an odd integer is odd.

### QUESTIONS TO PONDER

This is the first in a series of sections titled Questions to Ponder. In these sections you will find questions that you can play with at your leisure; you may work with other students in your class or challenge friends that are not in your class. Some of the questions should be resolvable with a bit of work. Some will become tractable as you proceed through the book. Some will be harder, and you may not be able to solve them completely, but I will only include such problems if you can make some progress on them at least by looking at examples. Some questions are philosophical in nature and their answers may be open-ended.

1. The following two statements were given as alternate definitions for an odd integer:

* An integer \( z \) is odd if it is not even.
* An integer \( z \) is odd if there exists an integer \( w \) so that \( z = 2w + 1 \).

One would hope that these definitions are equivalent. (That one is just a rephrasing of the other?) Can you prove this?

2. You should try to prove that \( \sqrt{2} \) is irrational. (Remember: A rational number is a number that can be written as a ratio of integers. One classic proof assumes \( \sqrt{2} \) is rational; that is, \( \sqrt{2} = \frac{a}{b} \). What can you say about the prime factors of \( m^2 \) and \( 2n^2 \) and what does this tell you?)

3. Try to prove that there are infinitely many prime numbers.

4. Try to prove that every positive number has a positive square root. (That is, prove that "For any positive real number \( x \) there exists a positive real number \( y \) such that \( y^2 = x \)."

5. Can every positive integer be written as the sum of distinct powers of two?

6. Can every even integer greater than 2 be expressed as the sum of two prime numbers? (This is the famous "Goldbach conjecture." The question was first asked in 1742. Mathematicians continue to struggle with it today. No one knows the answer.)

A set is a many that allows itself to be thought of as a one.—Cantor

### 2.1 Sets and Set Notation

Mathematical structures are built from sets. We could in principle begin with the axioms of set theory, deducing from them all of the useful properties of sets and constructing such familiar mathematical objects as real numbers, functions, and so on. We are not going to do this. A trip through axiomatic set theory is time-consuming and can sometimes obscure essentially simple ideas about sets. Furthermore, most mathematicians do their work without going all the way back to the underlying axioms of set theory. Intuitive ideas about sets and a few useful theorems (all of which are justifiable from the axioms) suffice for most purposes.

In this chapter, we will develop many of these set-theoretic ideas and theorems without a direct appeal to the axioms. We will assume without further comment the elementary arithmetic properties of the natural numbers:

\[ N = \{1, 2, 3, 4, \ldots \}. \]

In examples and exercises we will take advantage of the reader's informal understanding of the integers:

\[ \mathbb{Z} = \{0, \pm1, \pm2, \pm3, \pm4, \ldots \}. \]

the rational numbers \( \mathbb{Q} \) (quotients of integers), the real numbers \( \mathbb{R} \) (both rational and irrational numbers), and the Cartesian plane \( \mathbb{R}^2 \) (ordered pairs of real numbers).\(^1\)

\(^1\) Justifying these assumptions in retrospect, in Appendix A we set out some axiomatic set theory and build the natural numbers, and we give some indication of how to derive their fundamental arithmetic properties; divisibility and prime factorization are discussed in Chapter 6. In Chapter 8 we discuss the axioms for the real number system. In Appendix B we build the real numbers from set theory.