Prepare tabs

- (1) Announcements.
- (2) Graph $x^2 - y^2$. Shrink to 3d only.
- (3) Graph $x \sin y$. Shrink to contour only & hide address.
- (4) Hopf Video.

Show tab 1. Read announcements

Previously.

$\mathbb{R}^n = \text{ordered lists } (x_1, \ldots, x_n) \text{ of } n \text{ real numbers.}$

Today’s objectives.

- Match the equations, graphs and/or contour maps of functions on $\mathbb{R}^2$; sketch simple examples.
- Find the domain of a function.
Visualizing functions [14.1]

Let \( f \) be a function from \( \mathbb{R}^n \) to \( \mathbb{R} \).

Note: Written \( f : \mathbb{R}^n \rightarrow \mathbb{R} \).

- “:” = “is a function from”
- “→” = “to”

Today’s goal: Visualize \( f \).

Function of one variable.
Consider \( f : \mathbb{R} \rightarrow \mathbb{R} \). How do you visualize \( f \)?

The graph of \( f \) in \( \mathbb{R}^2 \) is the set
\[
\{y = f(x)\}.
\]

It’s a curve in \( \mathbb{R}^2 \); it could be made of wire.

Example. \( f(x) = x^2 \).

Function of two variables.
Consider \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \). How can we visualize \( f \)?

Example. \( f(x, y) = x^2 - y^2 \)

Method I: The graph of \( f \) is the set
\[
\{z = f(x, y)\}.
\]

It’s a surface in \( \mathbb{R}^3 \); it could be made out of sheet metal.
Question 1. Can you draw (or visualize) \( z = x^2 - y^2 \)?

(A) Yes, I have.
(B) Not yet.

Everyone click (B) now. Then click (A) when you can draw (or visualize) it.

Step I: Find it’s intersection with planes.

Intersect with \( y = 0 \)

\[
2 = x^2 - 0^2 = x^2
\]

Intersect with \( x = 0 \).

\[
2 = 0^2 - y^2 = -y^2
\]

Intersect with \( y = k \)

\[
2 = x^2 - k^2
\]

Intersect with \( x = k \)

\[
2 = k^2 - y^2
\]

\[ z = k \]

\[
\begin{cases}
K = 0: \ (x+y)(x-y) \\
K = 1: \ x^2 + y^2 = 1 \\
K = -1: \ x^2 - y^2 = -1
\end{cases}
\]

Ask students to take a minute to try.
Extra tricks:

Find the limit of \( f(x, y) = x^2 - y^2 \) as \( x \to \pm\infty \) and \( y \to \pm\infty \).

\[
\lim_{x \to \pm\infty} x^2 - y^2 = +\infty \quad \lim_{y \to \pm\infty} x^2 - y^2 = -\infty
\]

Is \( f \) periodic? For example, is \( f(x + \pi, y) = f(x, y) \)?

\( \backslash N \backslash o \)

Look for symmetry.

\[
\begin{aligned}
  f(x, y) &= f(-x, y) = f(x, -y) = f(-x, -y), \\
\end{aligned}
\]

\( \{z = x^2 - y^2\} \) is a \textbf{hyperbolic paraboloid}.

\( (0, 0) \) is a \textbf{saddle point} of \( f(x, y) = x^2 - y^2 \).

**Method II:** The \textbf{level set} of \( f \) at \( k \in \mathbb{R} \) is the set

\[ \{ f(x, y) = k \} \]

- It’s a curve in \( \mathbb{R}^2 \).
- It’s the intersection of the graph \( \{z = f(x, y)\} \) with \( \{z = k\} \).

A \textbf{contour map} for \( f \) is the union of level sets \( \{ f(x, y) = k \} \) for regular choices of \( k \), e.g., for all integers.

**Example:** Topographical maps are contour maps.

**Example:** Contour map for \( f(x, y) = x^2 - y^2 \).

Show tab (2) & switch to contour.
Claim: The slope of \( f \) is steeper where lines are closer together.

**Question 2.** Show tab (3).

Where is the slope steepest?

(A) \((0, 0)\)
(B) \((-1, 0)\)
(C) \((0.5, 0)\)
(D) \((1, 1.5)\)
(E) \((0, 1.5)\)

Given a function:

- the **domain** is the set of points where it’s defined, and
- the **range** is the set of values attained.

**Example.** Find the domain and range of \( f(x, y) = \sqrt{4 - x^2 - y^2} \).

**Solution.**

\[
\begin{align*}
\text{Domain: } & \{ x^2 + y^2 \leq 4 \} \\
\text{Range: } & [0, 2]
\end{align*}
\]

Check with your neighbor. Do the pictures agree?

(A) Yes
(B) No

**Solution.**

\( z = x^2 + y^2 \) is an **elliptic paraboloid**.
Question 4. Which method would you prefer:

(1) Writing on blackboard.
(2) Fill-in on ipad.
(3) I’d like to have some lectures on the blackboard before deciding.
(4) Other

Three variables. Consider $f : \mathbb{R}^3 \rightarrow \mathbb{R}$.

Example. Let $f(x, y, z) = x^2 + y^2 + z^2$.

Can you visualize the graph?

Only method: The level set of $f$ at $k \in \mathbb{R}$ is the set

$$\{ f(x, y, z) = k \}.$$

It’s a surface in $\mathbb{R}^3$.

Example. Find the level sets of $f(x, y, z) = x^2 + y^2 + z^2$.

Cool Math.

Let $S^n := \{|x| = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$.

The Hopf fibration is a map $f : S^3 \rightarrow S^2$.

How do we visualize $f$?

$S^3$ minus one point is $\mathbb{R}^3$, and so we restrict to $\mathbb{R}^3$.

Given $(a, b, c) \in S^2$, the level set $f(x, y, z) = (a, b, c)$ is a curve in $\mathbb{R}^3$. Show video