Prepare tabs:
(1) Graph $e^{-t} \sin x$, $x = 0$ to $4 \pi$, $t = 0$ to $10$; switch to contour plot & hide address.
(2) Announcements.
(3) Graph $\frac{xy}{x^2+y^2}$
(4) lecture notes.

Show tab 1 & write question on board.

**PREVIOUSLY**

Note: dark = negative & light = positive.

**Question 1.**

*When $x = 2$ & $t = 4$ ...*

(A) $f_t > 0$ & $f_{xx} > 0$.
(B) $f_t > 0$ & $f_{xx} < 0$.
(C) $f_t < 0$ & $f_{xx} > 0$.
(D) $f_t < 0$ & $f_{xx} = 0$.
(E) $f_t < 0$ & $f_{xx} < 0$.

Show tab 2 & read announcements.
**PDE’s**

**ODE:** Ordinary differential equation
One independent variable

**Example.**

\[ p(t) = \text{population at time } t. \]

\[ p'(t) = cp(t) \]

\[ \Rightarrow p(t) = p_0e^{ct}. \]

**PDE:** Partial differential equation
More variables

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Draw rod w/ snowflake at both end and fire in the middle.

What will happen?

\[ u(x, t) = \text{temperature of rod at} \]

• position \( x \) &

• time \( t \).

Heat equation:

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \]
Say \( u(x, 0) = \sin x \).

What do we expect?

Graph \( \sin x \) between 0 and \( 4\pi \).

Where \( \frac{\partial^2 u}{\partial x^2} > 0 \), temp increases. Draw arrows to these regions.

Where \( \frac{\partial^2 u}{\partial x^2} < 0 \), temp decreases. Draw arrows to these regions.

**Example.** Show \( u = e^{-t} \sin x \) satisfies the heat equation,

\[
\begin{align*}
    u_t &= -e^{-t} \sin x \\
    u_x &= e^{-t} \cos x \\
    u_{xx} &= -e^{-t} \sin x \\
    \Rightarrow u_t &= u_{xx}.
\end{align*}
\]

Show tab 1.

Note: You do not have to find solutions to PDE’s, just check if functions are solutions.
The linearization of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(x_0, y_0)$ is

$$L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Note:

- $L$ is a linear function.
- $L(x_0, y_0) = f(x_0, y_0)$.
- $\frac{\partial L}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)$.
- $\frac{\partial L}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$.

$\Rightarrow L$ the linear function that’s most like $f$ near $(x_0, y_0)$.

For $x$ and $y$ near $(x_0, y_0)$

$$f(x, y) \simeq L(x, y).$$

This is the linear approximation.
Example.

Find the linearization of \( f(x, y) = \frac{x}{x+y} \) at \((1, 1)\).

\[
f(1, 1) = \frac{1}{2}
\]

\[
f_x = \frac{1}{x+y} - \frac{x}{(x+y)^2}
\]

\[
\Rightarrow f_x(1, 1) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.
\]

\[
f_y = -\frac{x}{(x+y)^2}
\]

\[
\Rightarrow f_y(1, 1) = -\frac{1}{4}
\]

\[
\Rightarrow L(x, y) = \frac{1}{2} + \frac{1}{4}(x-1) - \frac{1}{4}(y-1)
\]

Question 2.

Use \( L \) to estimate \( f(1.1, .8) \)

(A) .45
(B) .5
(C) .55
(D) .575
(E) .6
In fact, \( f(1.1, .8) = 5.79 = + \).

The error in the linear approx. at \((x_0 + \Delta x, y_0 + \Delta y)\) is
\[
E(\Delta x, \Delta y) := f(x_0 + \Delta x, y_0 + \Delta y) - L(x_0 + \Delta x, y_0 + \Delta y).
\]

We say \( f \) is **differentiable** at \((x_0, y_0)\) \(\iff\)
\[
\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{E(\Delta x, \Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0.
\]

\(\iff\) If we zoom in on the graph of \( f \), we get the graph of \( L \)
\(\iff\) \( L \) is a good approximation to \( f \).

**Theorem.** If \( f_x \) & \( f_y \) exist near \((a, b)\) & are continuous
at \((a, b)\), then \( f \) is differentiable at \((a, b)\).

**Example.**
Is \( f(x, y) = \frac{x}{x+y} \) differentiable at \((1, 1)\)?

Yes, because \( f_x \) and \( f_y \) are continuous near \((1, 1)\).

Warning: may fail if \( f_x \) & \( f_y \) exist but aren’t continuous.

If \( \sim \) 15 minutes remaining, skip example.

**Example.** Let
\[
f(x, y) := \begin{cases}
\frac{x^2y^2}{(x^2+y^2)^2} & (x, y) \neq (0, 0) \\
0 & (x, y) = (0, 0)
\end{cases}
\]
\[
f(x, 0) = f(0, y) = 0 \Rightarrow f_x(0, 0) = f_y(0, 0) = 0 \text{ exist.}
\]
However, \( \lim_{(x,y) \to (0,0)} f(x, y) \) doesn’t exist,
\(\Rightarrow f \) isn’t continuous or differentiable at \((0, 0)\). **Show tab 3.**
Tangent planes.

Let $z_0 = f(x_0, y_0)$. The **tangent plane** to $f$ at $(x_0, y_0, z_0)$ is the graph of the linear approx. to $f$ at $(x_0, y_0)$.

$$z = z_0 + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

**Example.**

*Find the tangent plane to $f(x, y) = \frac{x}{x+y}$ at $(1, 1, \frac{1}{2})$.*

$$z = \frac{1}{2} + \frac{1}{4}(x - 1) - \frac{1}{4}(y - 1)$$

Note:

- The curve $x = x_0 \land z = f(x, y)$ lies in the graph of $f$.
- It's tangent line at $(y_0, z_0)$;
  $$y \mapsto \left( x_0, y, z_0 + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \right),$$
  lies in the tangent plane to $f$ at $(x_0, y_0, z_0)$.
- So does the tangent line to $y = y_0 \land z = f(x, y)$.

Later, we’ll see the tangent plane at $(x_0, y_0, z_0)$ is the union of all tangent lines to curves in $z = f(x, y)$ containing $(x_0, y_0, z_0)$. 
Differentials.

The **differential** of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(x_0, y_0)$ is

$$df = \frac{\partial f}{\partial x}(x_0, y_0)dx + \frac{\partial f}{\partial y}(x_0, y_0)dy.$$

Here,

- $dx$ and $dy$ are independent variables;
- they represent the change in $x$ and $y$, respectively.

**Question 3.**

*Find the tangent plane to $f(x, y) = 1 + ye^x$ at $(0, 2, 3)$.*

(A) $z = 3 + ye^x(x - 0) + e^y(y - 2)$

(B) $z = 2x + y$

(C) $z = 1 + 2x + y$

(D) $z = 2(x - 0) + 1(y - 2)$

(E) $z + 2(x - 0) + (y - 2) = 3$