Prepare tabs:

1. Graph $x^2y^2/(x^2 + y^2)^2$.
2. Announcements.
3. Graph $xy/(x^2 + y^4)^5$.
4. Graph $xy^2/(x^2 + y^4)$.
5. Lecture notes.

Previously

Show tab 1.

Write Question on board before class.

**Question 1.**

*Draw the contour plot of $f$.*

*(Hint: Start w/ $z = 0$ and $z = \frac{1}{4}$.)*

*Do you have a rough sketch?*

(A) Yes  
(B) Not Yet

If you’re done, check w/ your neighbor.

Show tab 2 & read announcements.
1. **Limits in several variables** [14.2]

Suppose we:

- **Want** a square w/ area 1.
- **Get** a rectangle w/ side lengths $1 + x$ & $1 + y$.

Draw rectangle with side lengths $1 + x$ & $1 + y$ and shade error.

Error = area - 1
\[ E(x, y) = (1 + x)(1 + y) - 1 = xy + x + y. \]

We say \( \lim_{(x,y) \to (0,0)} E(x, y) = 0 \) \iff

Given any \( \epsilon > 0 \), we can find \( \delta > 0 \) so that
\[
0 < \sqrt{x^2 + y^2} < \delta \Rightarrow |E(x, y)| < \epsilon.
\]

Note: \( \sqrt{x^2 + y^2} < \delta \Rightarrow |x| < \delta \) \& \( |y| < \delta \).

Given \( \epsilon = \frac{1}{10} \)? Let \( \delta = \frac{1}{30} \).

\[
\sqrt{x^2 + y^2} < \frac{1}{30} \Rightarrow |E(x, y)| = |xy + x + y| \leq |x||y| + |x| + |y| \leq \frac{1}{30 \cdot 30} + \frac{1}{30} + \frac{1}{30} < \frac{1}{10} = \epsilon.
\]
Given any small $\epsilon > 0$. Let $\delta = \frac{\epsilon}{3}$.

$\sqrt{x^2 + y^2} < \frac{\epsilon}{3} \Rightarrow$

$$|E(x, y)| \leq |x||y| + |x| + |y| \leq \frac{\epsilon^2}{9} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon.$$ 

$\Rightarrow \lim_{(x, y) \to (0, 0)} E(x, y) = 0.$

We say $\lim_{(x, y) \to (a, b)} f(x, y) = L \iff$

Given $\epsilon > 0$, we can find $\delta > 0$ so that

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \Rightarrow |f(x, y) - L| < \epsilon$$

Example.

Let $z = f(x, y) = \frac{x^2y^2}{(x^2+y^2)^2}$.

• $y = 0 \Rightarrow z = \frac{x^2 \cdot 0^2}{(x^2+0^2)^2} = 0$

• $x = 0 \Rightarrow z = \frac{0^2 \cdot y^2}{(0^2+y^2)^2} = 0$

• $y = x \Rightarrow z = \frac{x^2 \cdot x^2}{(x^2+x^2)^2} = \frac{1}{4} \neq 0$. 

Draw & label lines.
We will use this question twice.

**Question 2.**

Find \( \lim_{(x,y) \to (0,0)} f(x, y) \).

(A) 0  
(B) \( \frac{1}{4} \)  
(C) 1  
(D) Another number  
(E) Does not exist
Polar coordinates: \( x = r \cos \theta \) & \( y = r \sin \theta \).

\[
f(x, y) = \frac{(r \cos \theta)^2(r \sin \theta)^2}{(r^2 \cos \theta^2 + r^2 \sin \theta^2)^2} = (\cos \theta \sin \theta)^2 = \frac{1}{4} \sin^2(2\theta).
\]

\Rightarrow \text{Any (punctured) disk around } (0, 0) \text{ includes } (x_0, y_0) & (x_1, y_1) \text{ w/ } f(x_0, y_0) = 0 \& f(x_1, y_1) = \frac{1}{4}.

\textbf{Theorem. If}

- \( f(x, y) \rightarrow L_1 \) as \( (x, y) \rightarrow (a, b) \) along curve \( C_1 \)
- \( f(x, y) \rightarrow L_2 \) as \( (x, y) \rightarrow (a, b) \) along curve \( C_2 \) &
- \( L_1 \neq L_2 \).

Then \( \lim_{(x,y) \rightarrow (a,b)} f(x, y) \) does not exist.

Note: the same conclusion holds if \( L_1 \) or \( L_2 \) does not exist.
Example.
Let \( f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}} \).

\[
\frac{|x|}{x^2 + y^2} \leq 1
\]
\[
\Rightarrow \frac{|xy|}{\sqrt{x^2 + y^2}} \leq |y|
\]
\[
\Rightarrow -|y| \leq \frac{xy}{\sqrt{x^2 + y^2}} \leq |y|
\]

Moreover,

\[
\lim_{(x,y) \to (0,0)} -|y| = \lim_{(x,y) \to (0,0)} |y| = 0.
\]

Reuse previous question.

**Question 3.** Find \( \lim_{(x,y) \to (0,0)} f(x, y) \).

(A) 0
(B) \( \frac{1}{4} \)
(C) 1
(D) Another number
(E) Does not exist
Squeeze Theorem:

If

- \( f(x, y) \leq g(x, y) \leq h(x, y) \) near \((a, b)\)
- \( \lim_{(x,y) \to (a,b)} f(x, y) = \lim_{(x,y) \to (a,b)} h(x, y) = L. \)

Then \( \lim_{(x,y) \to (a,b)} g(x, y) = L. \)

Polar coordinates:

\[
 f(x, y) = \frac{(r \cos \theta)(r \sin \theta)}{\sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta}} = \frac{r \cos \theta \sin \theta}{r} = \frac{1}{2} \sin 2\theta \\
\lim_{r \to 0} \frac{r}{2} \sin 2\theta = 0
\]

Show tab 3.

If time permits, show tab 5 & do this hard question. Leave time to explain answer.

**Question 4.**

\[
\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} \ldots
\]

(A) \( \ldots \text{doesn’t exist because the limits as } (x, y) \to (0, 0) \text{ along the lines } x = 0 \text{ & } x = y \text{ are different.} \)

(B) \( \ldots \text{exists because the limit as } (x, y) \to (0, 0) \text{ along any line } ax = by \text{ is } 0. \)

(C) \( \ldots \text{doesn’t exist because the limits as } (x, y) \to (0, 0) \text{ along the curves } y = x^2 \text{ & } x = y^2 \text{ are different.} \)

(D) \( \ldots \text{doesn’t exist by the Squeeze theorem.} \)

(E) \( \ldots \text{exists by the Squeeze theorem.} \)

After given explanation, show tab 4.