PREVIOUSLY

Let $S = \{3x + 2y + z = 1, x \geq 0, y \geq 0 \text{ & } z \geq 0\}$, oriented upward & $\mathbf{F}(x, y, z) = \langle 1, 0, 1 \rangle$.

Draw $S$.

$S$ is parameterized by

$$\mathbf{r}(u, v) = (u, v, 1 - 3u - 2v),$$

$$(u, v) \in D = \{u \geq 0, \ v \geq 0, \ 3u + 2v \leq 1\}.$$

**Theorem.** If $\mathbf{r}_u \times \mathbf{r}_v$ is positive,

$$\int \int_S \mathbf{F} \cdot dS := \int \int_S \mathbf{F} \cdot \mathbf{n} \ dS = \int \int_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \ dA$$

**Question 1.** Find $\int \int_S \mathbf{F} \cdot dS$.

(A) $\frac{1}{12}$
(B) $\frac{1}{6}$
(C) $\frac{1}{3}$
(D) $-\frac{1}{6}$
(E) $-\frac{1}{3}$

Save the answer in box. Show tab 2 & read announcements.
### 16.8 Stokes’ Theorem

Let $S$ be an oriented surface.

The **boundary curve** $\partial S$ of $S$ is the set of points that

- nearby, look like $\{y \geq 0\} \subset \mathbb{R}^2$ (instead of $\mathbb{R}^2$) &
- would be sharp if $S$ was made of metal.

**Point your head towards $\mathbf{n}$.**

**Orient $\partial S$ so that $S$ is on your left.**

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Draw a blob in $\mathbb{R}^2$ with a hole.
The outer circle is oriented counterclockwise, the inner circle is oriented clockwise.

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Draw a vertical cylinder, oriented outwards.
Looked at from top, the top circle is oriented clockwise, the bottom circle is oriented counterclockwise.
For today,

- \( S \) is “nice” \iff
  - \( S \) is piecewise smooth &
  - \( \partial S \) is one or more simple closed paths.
- \( F: S \to \mathbb{R}^3 \) has continuous 1st partial derivatives &

**Theorem** (Stokes’ theorem).

\[
\iint_S (\text{curl } F) \cdot d\mathbf{S} = \oint_{\partial S} F \cdot d\mathbf{r}
\]

Use this to integrate curl \( F \) over \( S \), if \( \partial S \) is simpler; or to integrate \( F \) over \( \partial S \), if curl \( F \) is simpler.

There’s a derivative on left & boundary on right, c.f., the Fundamental Theorems & Green’s theorem.

**Why?** Fundamental Theorem of Calculus
Example. Let $\mathbf{F} = \langle 1, x + \sin y^2, y - e^{z^3} \rangle$, $S = \{3x + 2y + z = 1, x \geq 0, y \geq 0 \& z \geq 0\}$, oriented upward & $C = \partial S$. Find $\int_C \mathbf{F} \cdot \mathbf{dr}$.

Draw $S$ & $C$; as appropriate, draw $D$.

Parameterize $S$:

$$\mathbf{r}(u, v) = \langle u, v, 1 - 3u - 2v \rangle,$$

$$(u, v) \in D = \{u \geq 0, v \geq 0 \& 1 - 3u - 2v \geq 0\}.$$

$$\mathbf{r}_u = \langle 1, 0, -3 \rangle \& \mathbf{r}_v = \langle 0, 1, -2 \rangle$$

$$\Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix}
i & j & k \\
1 & 0 & -3 \\
0 & 1 & -2
\end{vmatrix} = \langle 3, 2, 1 \rangle.$$

Since $1 > 0$, $\mathbf{r}_u \times \mathbf{r}_v$ points up. √

curl $\mathbf{F} = \begin{vmatrix}
i & j & k \\
\partial_x & \partial_y & \partial_z \\
1 & x + \sin y^2 & y - e^{z^3}
\end{vmatrix}
= \langle 1 - 0, 0 - 0, 1 - 0 \rangle = \langle 1, 0, 1 \rangle.$$

By Stokes’ Theorem,

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{dr} = \iint_S \text{curl} \mathbf{F} \cdot \mathbf{dS} = \iint_D \langle 1, 0, 1 \rangle \cdot \langle 3, 2, 1 \rangle \, dA$$
$$= \iint_D (3 + 0 + 1) \, dA = 4(\text{area of } D) = 4 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3}.$$
Example. Let \( F = \langle y, -x, xe^{y^2} \rangle \) & \( S = \{ z = 10 - x^2 - y^2 \ \& \ z \geq 1 \}, \) oriented upwards. 
Find \( \iint_S \text{curl} \ F \cdot dS \)

\[ \partial S = \{ x^2 + y^2 = 9 \ \& \ z = 1 \}, \) oriented counterclockwise & parameterized by \( r(t) = \langle 3 \cos t, 3 \sin t, 1 \rangle, \) \( 0 \leq t \leq 2\pi. \)

By Stokes’ Theorem,

\[
\iint_S \text{curl} \ F \cdot dS = \oint_{\partial S} F \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t) \, dt
\]

\[
= \int_0^{2\pi} \langle 3 \sin t, -3 \cos t, 3 \cos t e^{9\sin^2 t} \rangle \cdot \langle -3 \sin t, 3 \cos t, 0 \rangle \, dt
\]

\[
= \int_0^{2\pi} (-9 \sin^2 t - 9 \cos^2 t) \, dt = \int_0^{2\pi} -9 \, dt = -18\pi.
\]

If < 10 minutes remain, skip to subsection “Curl”.

Example. Assume \( F = \langle P(x, y), Q(x, y), 0 \rangle \) & \( S \subset \{ z = 0 \}, \) oriented upwards.

\[
\int_{\partial S} P \, dx + Q \, dy = \oint_{\partial S} F \cdot dr = \iint_S (\text{curl} \ F) \cdot dS
\]

\[
= \iint_S (\text{curl} \ F) \cdot n \, dS = \iint_S \langle 0, 0, Q_y - P_x \rangle \cdot k \, dS
\]

\[ = \iint_S (Q_y - P_x) \, dA. \]

This is Green’s theorem!
Question 2. Let \( F = \langle x \sin z, y \sin z, e^{x+y} \rangle \) & \( S = \{ x^2 + y^2 + z^2 = 9 \} \), oriented outwards. Find \( \iint_S \text{curl} \ F \cdot d\mathbf{S} \). 

(A) \(-12\pi\).
(B) \(-9\).
(C) 0.
(D) 9.
(E) 12\pi.

Curl.

Let \( C \) be an oriented curve w/ unit tangent vector \( \mathbf{T} \). Recall: The circulation of \( F \) around \( C \) is

\[
\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r}.
\]

It measures if fluid flows with \( C \) or against \( C \). Look so that a unit vector \( \mathbf{v} \) points towards you. Place a tiny paddle wheel on a disk \( S \) with normal \( \mathbf{v} \). By Stokes’ Theorem, the circulation around \( \partial S \) is \( \iint_S \text{curl} \ F \cdot \mathbf{n} \, dS \), so that paddlewheel...

- Rotates counterclockwise \( \iff \) curl \( \mathbf{F} \cdot \mathbf{v} > 0 \) &
- Rotates clockwise \( \iff \) curl \( \mathbf{F} \cdot \mathbf{v} < 0 \).
- \( |\text{curl} \ F \cdot \mathbf{v}| \) gives speed of rotation \( \Rightarrow \)
  - Doesn’t rotate \( \iff \) curl \( \mathbf{F} \cdot \mathbf{v} = 0 \).
  - The fastest rotation is when \( \mathbf{v} = \pm \text{curl} \ F \).