Prepare tabs:
(1) Lecture notes
(2) Announcements
(3) Plot $xy = 1$, $xy = 2$, $xy^2 = -1$ & $xy^2 = -2$.

Show tab 1.

PREVIOUSLY

For today, assume

- all regions are “nice” $\Leftrightarrow$ we know how to integrate.

$T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear $\Leftrightarrow$ $T(u, v) = (au + bv, cu + dv)$.

**Theorem.** Fix $D \subset \mathbb{R}^2$ & $f: T(D) \to \mathbb{R}$ continuous,

\[
\iint_{T(D)} f(x, y) \, dx \, dy = \iint_D f(T(u, v)) |ad - bc| \, du \, dv
\]

**Question 1.** Given $\alpha, \beta > 0$,

let $D = \{u^2 + v^2 \leq 1\}$ & $B = \left\{ \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} \leq 1 \right\}$.

Draw $D$ on left & $B$ on right; when appropriate, draw $T$.

Find the area of $B$. Hint: Don’t integrate!

Find a linear transformation $T$ with $T(D) = B$.

(A) $\frac{1}{\alpha \beta \pi}$

(B) $\pi$

(C) $\alpha \beta \pi$

(D) $\alpha \pi$.

Show tab 2 & read announcements.
General change of variables [15.10]

Let $f, g: \mathbb{R}^2 \to \mathbb{R}$ have continuous 1st partial derivatives.

Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(u, v) = (g(u, v), h(u, v)) \iff x = g(u, v) \& y = h(u, v)$$

The image of $D \subset \mathbb{R}^2$ is

$$T(D) = \{T(x, y) \mid (x, y) \in D\}.$$  

We say $T: D \to \mathbb{R}^2$ is one-to-one if no two points have the same image.
Example. Define $T: D \rightarrow \mathbb{R}^2$ by $T(r, \theta) = (r \cos \theta, r \sin \theta)$.

Draw ($r, \theta$) plane on left, ($x, y$) plane on right, & indicate $T$.

1. $D = [0, \infty) \times (-\infty, \infty)$
   \[ T(1, 0) = T(1, 2\pi) = T(1, 4\pi) = \ldots \]
   \[ T(0, 0) = T(0, \pi/2) = T(0, \pi) = \ldots \]
   $\Rightarrow$ $T$ is not one-to-one

2. $D = (0, \infty) \times [0, 2\pi)$
   $\Rightarrow$ $T$ is one-to-one.

3. $D = [0, \infty) \times [0, 2\pi]$
   $\Rightarrow$ $T$ is one-to-one except on the boundary of $D$.

The Jacobian of $T$ is

\[
\frac{\partial(x, y)}{\partial(u, v)} := \left| \begin{array}{cc}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{array} \right|
\]

Example.

\[
\frac{\partial x}{\partial r} = \cos \theta \quad \frac{\partial x}{\partial \theta} = -r \sin \theta
\]
\[
\frac{\partial y}{\partial r} = \sin \theta \quad \frac{\partial y}{\partial \theta} = r \cos \theta
\]

\[
\frac{\partial(x, y)}{\partial(r, \theta)} = \left| \begin{array}{cc}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{array} \right| = r \cos^2 \theta + r \sin^2 \theta = r
\]
Question 2.
Let $x = u^2/v$ & $y = v/u$ & find $\frac{\partial (x,y)}{\partial (u,v)}$.

(A) I’m working on it.
(B) I have an answer, but don’t agree w/ my neighbor.
(C) We agree!

Theorem. Fix $D \subset \mathbb{R}^2$ & $f : T(D) \to \mathbb{R}$ continuous. Assume $T : D \to \mathbb{R}^2$ is one-to-one except possibly on its boundary & $\frac{\partial (x,y)}{\partial (u,v)} \neq 0$.

\[
\int\int_{T(D)} f(x,y) \, dx \, dy = \int\int_D f(T(u,v)) \left| \frac{\partial (x,y)}{\partial (u,v)} \right| \, du \, dv
\]

Why?
Consider the rectangle $\Box = \{ u_0 \leq u \leq u_0 + \Delta u, \quad v_0 \leq v \leq v_0 + \Delta v \}$

Near $(u_0, v_0) \in D$, $f$ is approximately linear.

$\Rightarrow$ for small $\Delta u$ & $\Delta v$,

$T(\Box) \simeq$ parallelogram w/ edges $\Delta u \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \rangle$ & $\Delta v \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \rangle$

Draw $T(\Box)$ and the parallelogram.

$\Rightarrow \text{Area}(T(\Box)) \simeq \left| \frac{\partial (x,y)}{\partial (u,v)} \right| \Delta u \Delta v$
**Example.** Let $B$ be bounded by:

$xy = 1$, $xy = 2$, $xy^2 = -1$ & $xy^2 = -2$

Find $\iint_B y^2 dA$

Draw $B$ & show tab 3.

- Let $u = xy$ & $v = xy^2 \Leftrightarrow x = u^2/v$ & $y = v/u$.

- Let $D = \begin{cases} 1 \leq u \leq 2 \\ -2 \leq v \leq -1 \end{cases} \Rightarrow T(D) = B$.

- $f(x, y) = y^2 \Rightarrow f(T(u, v)) = v^2/u^2$.

- $\left| \frac{\partial (x, y)}{\partial (v, u)} \right| = \left| \frac{1}{v} \right| = \left| -\frac{1}{v} \right|$ since $v < 0$.

\[
\iint_B y^2 \, dA = \int_1^2 \int_{-2}^{-1} \left( \frac{v^2}{u^2} \right) \left( -\frac{1}{v} \right) \, dv \, du \\
= -\int_1^2 u^{-2} \, du \int_{-2}^{-1} v \, dv = u^{-1}\bigg|_1^2 \cdot v^2/2\bigg|_{-2}^{-1} \\
= (1/2 - 1)(1/2 - 4/2) = 3/4
\]
Three dimensions.

Given $g, h, k: \mathbb{R}^3 \to \mathbb{R}$, define $T: \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(u, v, w) = (g(u, v, w), h(u, v, w), k(u, v, w))$$

$$x = g(u, v, w), \quad y = h(u, v, w) \quad \& \quad z = k(u, v, w).$$

The Jacobian of $T$ is

$$\frac{\partial (x, y, z)}{\partial (u, v, w)} = \det \begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{pmatrix}$$

Theorem. Under similar conditions,

$$\iint_{T(D)} f(x, y, z) \, dx \, dy \, dz = \iiint_D f(T(u, v, w)) \left| \frac{\partial (x, y, z)}{\partial (u, v, w)} \right| \, du \, dv \, dw.$$
Why? Skip to examples if < 5 minutes remain.

Near any point, $T$ is approximately a linear function (+ a constant).

So we may assume

$$T(u, v, w) = (au + bv + cw, du + ev + fw, gu + hv + iw)$$

- $T(0, 0, 0) = (0, 0, 0)$
- $T(1, 0, 0) = (a, d, g)$
- $T(0, 1, 0) = (b, e, h)$
- $T(0, 0, 1) = (c, f, i)$

$\Rightarrow [0, 1] \times [0, 1] \times [0, 1] \mapsto$ parallelepiped with edges $\langle a, d, g \rangle, \langle b, e, h \rangle \& \langle c, f, i \rangle$

$\Rightarrow$ In terms of volume,

$$1 \mapsto \left| \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \right| = \left| \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \right| = \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

**Example. Cylindrical coordinates**

$$x = r \cos \theta, \quad y = r \sin \theta \quad \& \quad z = z \quad \Rightarrow \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$$

**Example. Spherical coordinates**

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad \& \quad z = \rho \cos \phi$$

$\Rightarrow \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \rho^2 \sin \phi$
Question 3. Calculate the Jacobian for the first/second example above. Do you get the right answer?

(A) No
(B) Yes
(C) I’m not done.