Prepare tabs:
   (1) Plot a vector field.
   (2) Announcements.

Show tab 1.

PREVIOUSLY

Question 1.

*Is $F$ conservative?*

(A) Yes.

(B) No.

(C) We don’t have enough information.
Recall:
Divide \([a, b]\) into \(n\) equal pieces of width \(\Delta x = \frac{b-a}{n}\).
Pick \(x_i^* \in [x_{i-1}, x_i]\) for all \(i\).
Given \(f: [a, b] \to \mathbb{R}\),
\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i^*) \Delta x,
\]
if the limit exists & doesn’t depend on the \(x_i^*\).

Draw picture of \([a, b]\) broken into 5 pieces at \(x_0, \ldots, x_5\), the graph of \(f\), and the approximating rectangles.
If \(f \geq 0\), \(\int_a^b f \, dx = \text{area over } [a, b] \& \text{under graph of } f\).
Claim: If \(f\) is continuous (except a finite number of jump discontinuities), the limit exists & doesn’t depend on the \(x_i^*\).
Fix a rectangle $D = [a, b] \times [c, d] = \left\{ \begin{array}{l} a \leq x \leq b \\ c \leq y \leq d \end{array} \right\}$.

Divide $[a, b]$ into $n$ equal pieces of width $\Delta x = \frac{b-a}{n}$ & $[c, d]$ into $m$ pieces of width $\Delta y = \frac{d-c}{m}$.

Pick $(x_{ij}^*, y_{ij}^*) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for all $i$ & $j$.

Draw rectangle broken into $3 \times 2$ pieces & add $(x_{ij}^*, y_{ij}^*)$

Given $f: D \to \mathbb{R}$, define

$$\iint_D f \, dA := \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*) \Delta x \Delta y,$$

if the limit exists and doesn’t depend on the $(x_{ij}^*, y_{ij}^*)$.

In this case, we say $f$ is integrable.

Geometric interpretations:

$$\iint_D f \, dA = \text{(average value of } f \text{ on } D)(\text{area of } D).$$

If $f \geq 0$,

$$\iint_D f \, dA = \text{volume over } D \& \text{ under the graph of } f.$$
We say $f: D \to \mathbb{R}$ is **bounded** if there exists $M \in \mathbb{R}$ so that $f(P) \leq M$ for all $P \in D$.

**Claim:** If $f: D \to \mathbb{R}$ is bounded and continuous except at a finite number of smooth curves, then $f$ is integrable.

For today: Assume all functions satisfy this condition.

Midpoint rule: Pick $x_{ij}^* = \frac{x_{i-1} + x_i}{2}$ & $y_{ij}^* = \frac{y_{i-1} + y_i}{2}$.

**Example.** Let $D = [0, 4] \times [1, 5]$ & $f(x, y) = x + y$. Use the midpoint rule w/ $n = m = 2$ to estimate $\int \int_D f \, dA$

Draw interval w/ midpoints.

$$\int \int_D f \, dA \simeq f(1, 2)4 + f(3, 2)4 + f(1, 4)4 + f(3, 4)4$$

$$= (3 + 5 + 5 + 7)4 = 80.$$
Question 2. Estimate the average value of $f$ on $D$.

(A) 0
(B) 5
(C) 10
(D) 20
(E) 80

How do we calculate $\iint_D f \, dA$?

**Theorem** (Fubini’s Theorem).

$$\iint_D f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy = \int_a^b \int_c^d f(x, y) \, dy \, dx$$

In practice:

1. Do inner integral first.
   - Treat outer variable as a constant.
2. Then do outer integral.

**Example.** Calculate the integral we estimated earlier.

$$\int_0^4 (x + y) \, dx = \left( \frac{x^2}{2} + yx \right) \bigg|_0^4 = 8 + 4y$$

$$\Rightarrow \iint_D f \, dA = \int_1^5 \int_0^4 (x + y) \, dx \, dy = \int_1^5 (8 + 4y) \, dy$$

$$= (8y + 2y^2) \bigg|_1^5 = 40 + 50 - 8 - 2 = 80.$$
Why is Fubini’s Theorem true?

Assume $f \geq 0$.

Let $S$ be the region over $D$ & under graph of $f$.

Draw region over $S$ & under graph of $f$.

Given $x_0 \in \mathbb{R}$, the area of $S \cap \{x = x_0\}$ is

$$A(x_0) = \int_c^d f(x_0, y) \, dy.$$ 

Note: $\cap$ = “intersection”; points which lie in both sets.

$$\int \int_D f \, dA = \text{volume over } D \ & \text{under graph of } f$$

$$= \int_a^b A(x) \, dx = \int_a^b \int_c^d f(x, y) \, dy \, dx$$
If time permits, ask them the question or do it yourself

**Question 3.**

Let \( D = [0, 2] \times [-3, 1] \) and \( f(x, y) = 3x^2 + 3y^2 \).

Find \( \iint_D f \, dA \).

(A) \(-12\)
(B) 14
(C) 28
(D) 42
(E) 88