Prepare tabs:
(1) Plot $\langle -y, x \rangle$.
(2) announcements.
(3) Plot $\frac{1}{x^2+y^2}(-y, x)$.

Previously

**Question 1.**
$C_1$ is $x^2 + y^2 = 1$ & $x \leq 0$ and $C_2$ is $x^2 + y^2 = 1$ & $x \geq 0$.
Orient both from $(-1, 0)$ to $(1, 0)$ & let $F = \langle -y, x \rangle$.
Use $\int_C F \cdot dr = \int_C F \cdot Tds$ to find $\int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr$.

(A) $-2\pi$
(B) $-\pi$
(C) 0
(D) $\pi$
(E) $2\pi$

Show tab 1.

If you are done: answer the same question for

$$F = \begin{pmatrix} -y \\ x^2 + y^2 \end{pmatrix}.$$
Conservative vector fields I [16.3]

Let $\mathbf{F}$ be a vector field on $D \subset \mathbb{R}^n$.

We say $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independant** if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

if $C_1$ & $C_2$ start at the same point & end at the same point.

**Example.** $\langle -y, x \rangle$ is not path independant.

**Example.**

On $x^2 + y^2 = 1$, $\frac{1}{x^2+y^2} \langle -y, x \rangle = \langle -y, x \rangle \Rightarrow$

$$\int_{C_i} \frac{1}{x^2+y^2} \langle -y, x \rangle \cdot d\mathbf{r} = \int_{C_i} \langle -y, x \rangle \, dr \Rightarrow$$

$$\frac{1}{x^2+y^2} \langle -y, x \rangle$$

is not path independant.

Show tab 3.

Let $\mathbf{r}(t), a \leq t \leq b$, parameterize a curve $C$.
We say $C$ is **closed** if it begins & ends at the same point $\iff \mathbf{r}(a) = \mathbf{r}(b)$.

Draw two curves that are closed (one of which intersects itself) and one that isn’t.

Label $\mathbf{r}(a)$ & $\mathbf{r}(b)$ indicate which are closed.
Theorem.

\[ \int_C \mathbf{F} \, d\mathbf{r} \text{ is path independant} \iff \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for every closed curve } C. \]

Proof.
Assume \( \mathbf{F} \) is path independant.
Let \( C \subset D \) be a closed oriented curve.

Draw & label \( C \). Add & label \( P, Q \) & \( C_i \) when appropriate.

Pick \( P \) & \( Q \) on \( C \).
Let \( C_1 \subset C \) be the path from \( P \) to \( Q \) & \( C_2 \) from \( Q \) to \( P \).
\( C_1 \) & \(-C_2\) both start at \( P \) & end at \( Q \)

\[ \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \]
\[ = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0 \]

To prove the converse,
join \( C_1 \) & \(-C_2\) into a closed path \( C \). \( \square \)
Let $C$ be $x^2 + y^2 = 1$, oriented clockwise. Recall that $\int_C \langle -y, x \rangle \cdot \mathrm{d}r = -2\pi$.

We say $D \subset \mathbb{R}^n$ is **open** if:

It doesn’t contain any of its boundary points. \\
$\Leftrightarrow$ There is a disk that lies in $D$ around each point in $D$.

In practice: Defined by $<, >$, and/or $\neq$ (not $\leq, \geq$, or $=$).

**Example.**

\[
\begin{align*}
  x^2 + y^2 &< 1 & x^2 + y^2 &< 1 & x^2 + y^2 &\leq 1 \\
  y &> 0 & y &\geq 0 & y &\geq 0
\end{align*}
\]

*Draw each region.*

**Boundary:**

$(x^2 + y^2 = 1 \land y \geq 1)$ and $(y = 0 \land -1 \leq x \leq 1)$.

*Draw boundary & label regions open, closed, or neither.*
We say $D \subset \mathbb{R}^n$ is **connected** if:
Any two points in $D$ can be joined by a path in $D$.

In practice: Only one piece – won’t fall apart.

Draw two boxes; one w/ a complicated connected region and one w/ two disks.

Let $\mathbf{F}$ be a vector field on $D \subset \mathbb{R}^n$.
Recall: $\mathbf{F}$ is **conservative** iff $\mathbf{F} = \nabla g$ for some $g: D \rightarrow \mathbb{R}$.

**Theorem** (Fundamental Theorem of Line Integrals).

Given $g: C \rightarrow \mathbb{R}$,

$$
\int_C \nabla g \cdot d\mathbf{r} = g(\mathbf{r}(b)) - g(\mathbf{r}(a))
\Rightarrow
$$

**Theorem.**

$\mathbf{F}$ conservative $\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent.

**Example.** $\langle -y, x \rangle \not\in \frac{1}{x^2+y^2}\langle -y, x \rangle$ are not conservative.

**Theorem.**

If $D \subset \mathbb{R}^n$ is open & connected

$$
\int_C \mathbf{F} \cdot d\mathbf{r} \text{ is path independent } \iff \mathbf{F} \text{ is conservative.}
$$

If < 15 minutes remaining, skip to last page.
Why?
Fix \( P \in D \).

Given \( x \in \mathbb{R}^n \), define \( g(x) = \int_C \mathbf{F} \cdot d\mathbf{r} \),
where \( C \) is a curve from \( P \) to \( x \) in \( D \).

This is well-defined because \( D \) is connected &
\( \int_C \mathbf{F} \cdot d\mathbf{r} \) is path independant.

Claim: \( \nabla g = \mathbf{F} \)

Note: Suppose we know that \( \mathbf{F} = \nabla f \).
Let \( g(x) = f(x) - f(P) \) for all \( x \in \mathbb{R}^n \).
Then \( \nabla g = \nabla f = \mathbf{F} \) & \( g(P) = 0 \).
If \( C \) is a curve from \( P \) to \( x \), then
\[
g(x) = g(x) - g(P) = \int_C \nabla g \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r}.
\]
In practice:
Let \( \mathbf{F} = \langle P, Q \rangle \) be conservative.
To find \( f \) such that \( \nabla f = \mathbf{F} \):

1. Find \( h(x, y) \) with \( \frac{\partial h}{\partial x} = P \).
2. Find \( g(y) \) with \( \frac{\partial}{\partial y}(h + g) = Q \iff \frac{\partial g}{\partial y} = Q - \frac{\partial h}{\partial y} \).
3. Let \( f(x, y) = h(x, y) + g(y) \).

**Example.** \( \mathbf{F} = \langle \sin y, x \cos y + 2y \rangle \) is conservative.

Find \( f \) so that \( \nabla f = \mathbf{F} \).

1. Want \( h(x, y) \) with \( \frac{\partial h}{\partial x} = \sin y \).
   \( h(x, y) = x \sin y \).
2. Want \( g(y) \) with \( \frac{\partial g}{\partial y} = x \cos y + 2y - \frac{\partial}{\partial y}(x \sin y) = 2y \).
   \( g(y) = y^2 \).
3. \( f(x, y) = x \sin y + y^2 \).

Give explanation if time permits.

Why does this work?

\( h \) exists by assumption.

Then \( \frac{\partial}{\partial x}(f - h) = P - \frac{\partial f}{\partial x} = 0 \Rightarrow \)
\( f(x, y) = h(x, y) = g(y) \) for some \( g \).