Given a smooth curve $C \subset \mathbb{R}^2$ and $f \geq 0$,
\[
\int_C f \, ds = \text{area of fence under graph of } f \text{ and over } C.
\]
If $f$ is the linear density of a wire shaped like $C$,
\[
\int_C f \, ds = \text{total mass of wire } \&
\left( \frac{\int_C x f \, ds}{\int_C f \, ds}, \frac{\int_C y f \, ds}{\int_C f \, ds} \right) = \text{center of mass}.
\]

**Question 1.**
Let $C$ be parameterized by $\mathbf{r}(t) = \langle 3t, 4t \rangle$, $0 \leq t \leq 1$.
If the density is $f = 1 + y$, the total mass of the wire is
(A) 0  
(B) 5  
(C) 10  
(D) 15  
(E) 20

If done: Decide if your answer is reasonable.
Vector fields [16.1]

A vector field on \( \mathbb{R}^2 \) is a function \( F: \mathbb{R}^2 \to \mathbb{R}^2 \), assigning a vector \( F(x, y) = \langle P, Q \rangle \) to each \( (x, y) \in \mathbb{R}^2 \).

Uses:

- Wind speed & direction (at each point).
- Force magnitude & direction.
- Electric & magnetic fields.

Example. \( F(x, y) = \langle -y, x \rangle \).

To visualize, draw \( F(x, y) \) w/ tail \( (x, y) \).
(Rescale arrows if needed.)

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A vector field on \( \mathbb{R}^3 \) is a function \( F: \mathbb{R}^3 \to \mathbb{R}^3 \).
Example. Gravity
The force a large mass $M$ at $(0, 0, 0)$ exerts on a small mass $m$ at $(x, y, z)$ is:

$$\mathbf{F}(x, y, z) = -\frac{MmG}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \langle x, y, z \rangle$$

$$\Rightarrow |\mathbf{F}| = \frac{MmG}{x^2 + y^2 + z^2} \quad (Newton’s Law)$$

Sketch cross section of vector field.

For several bodies, add vector fields.

Draw two bodies w/ masses $M_1$ & $M_2$. Draw resulting forces $\mathbf{F}_1$ & $\mathbf{F}_2$. Draw their sum $\mathbf{F}$.

Example. Gradients
Given $g: \mathbb{R}^3 \rightarrow \mathbb{R}$, let

$$\mathbf{F} = \nabla g = \langle g_x, g_y, g_z \rangle.$$

In this case, we say

- $g$ is a potential function for $\mathbf{F}$ (potential energy)
- $\mathbf{F}$ is conservative.
Example. Let \( g = \frac{MmG}{\sqrt{x^2+y^2+z^2}} \).

\[
g_x = -\frac{1}{2} \left(2x\right) \frac{Mmg}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \Rightarrow \\
\nabla g = \frac{-MmG}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \langle x, y, z \rangle.
\]

\( \Rightarrow \) Gravity is conservative w/ potential (energy) \( g \).

Remember that the dot product of two vectors is positive if the angel between them is acute, zero if they are perpendicular, and negative if the angle is obtuse.

Show tab ??.

**Question 2.** \( F \) is ...

(A) \( \langle y, x \rangle \).

(B) \( \langle 2 \sin y, y - 2 \rangle \).

(C) \( \langle x - 2, x + 1 \rangle \).

(D) \( \langle 1, \sin y \rangle \).

Sketch & label \( F \).
LINE INTEGRALS \[16.2\]

For today: curves are smooth & vector fields continous.

Let \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \), \( a \leq t \leq b \), parameterize \( C \).

The line integral of \( \mathbf{F} = \langle P, Q, R \rangle \) along \( C \) is

\[
\int_C \mathbf{F} \cdot d\mathbf{r} := \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt
\]

\[
= \int_a^b \langle P, Q, R \rangle \cdot \langle x', y', z' \rangle dt
\]

\[
= \int_a^b (P(\mathbf{r}(t))x'(t) + Q(\mathbf{r}(t))y'(t) + R(\mathbf{r}(t))z'(t))dt
\]

- We also write \( \int_C \mathbf{F} \cdot d\mathbf{r} \) as \( \int_C Pdx + Qdy + Rdz \).
- A similar formula holds for \( C \) in \( \mathbb{R}^2 \).

**Example.** Let \( \mathbf{r}(t) = \langle t, t^2 \rangle \), \( 0 \leq t \leq 1 \) \& \( \mathbf{F} = \langle y, x \rangle \)

If < 10 minutes remains, skip to answer.

Sketch curve on vector field already on board.

**Question 3.** Is \( \int_C \mathbf{F} \cdot d\mathbf{r} \)

- (A) \( > 0 \)
- (B) \( = 0 \)
- (C) \( < 0 \)
- (D) Can’t tell without computing it.
\[ \mathbf{r}' = \langle 1, 2t \rangle \]
\[ \mathbf{F}(\mathbf{r}(t)) = \langle t^2, t \rangle \]

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle t^2, t \rangle \cdot \langle 1, 2t \rangle dt = \int_0^1 (t^2 + 2t^2)dt = t^3 \bigg|_0^1 = 1 - 0 = 1
\]

For constant force \( \mathbf{F} \) & displacement \( \mathbf{d} \),

\[ \text{work} = \mathbf{F} \cdot \mathbf{d}. \]

Sketch person box & label vectors.

Rocket travels along \( \mathbf{r}(t), a \leq y \leq b. \)

Draw rocket ship flying off planet.
\textbf{Claim:} Total work done by gravity $\mathbf{F}$ is

\[
\text{Work} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_C \mathbf{F} \cdot d\mathbf{r}
\]

Give proof if time remains.

\textit{Proof.} Break $[a, b]$ into $n$ even segments $[t_{i-1}, t_i]$ of width $\Delta t$.

Work done from $\mathbf{r}(t_i)$ to $\mathbf{r}(t_{i+1})$

\[
\simeq \mathbf{F}(\mathbf{r}(t_i)) \cdot (\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)) = \mathbf{F}(\mathbf{r}(t_i)) \cdot (\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)) \simeq \mathbf{F}(\mathbf{r}(t_i)) \cdot \mathbf{r}'(t_i) \Delta t
\]

Sum up and take $n \to \infty$ to get

\[
\text{Work} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt.
\]

\qed