Prepare tabs:
   (1) Parametric plot from Question 1.
   (2) announcements.

Show tab 1.

Previously

Question 1.

Choose the parameterization.

(A) \( x = t \) \& \( y = 1 - \sin t \)
(B) \( x = t - \sin t \) \& \( y = -\sin t \).
(C) \( x = t - \sin t \) \& \( y = t - \cos t \).
(D) \( x = -\sin t \) \& \( y = 1 - \cos t \).
(E) \( x = t - \sin t \) \& \( 1 - \cos t \).
Example (Cycloid).

A wheel of radius 1 rolls at 1 radian/sec. 
\( \mathbf{r}(t) \) is the position of an LED on the rim at time \( t \).

The center starts at \((0, 1)\) & the LED at the bottom.

\[ \mathbf{r}(t) = \langle t, 1 \rangle + \langle -\sin t, -\cos t \rangle = \langle t - \sin t, 1 - \cos t \rangle. \]

Upside down, this curve has remarkable properties:
Tautochrome: same time to bottom.

Brachistchrone: Minimizes time from A to B.

This is an infinite dimensional min/max problem: ("Calculus of variations")
For today: Assume curves are smooth ⇔ \( r' \) is continuous and nonvanishing.

Integration w/ one variable: Fix \( g: [a, b] \to \mathbb{R} \).

Divide \([a, b]\) into \( n \) subintervals \([x_{i-1}, x_i]\) of size \( \Delta x = \frac{a-b}{n} \).
Pick any \( x_i^* \in [x_{i-1}, x_i] \) for all \( i \).
Define
\[
\int_a^b g(x) \, dx := \lim_{n \to \infty} \sum_{i=1}^n g(x_i^*) \Delta x,
\]
if the limit exists & doesn’t depend on the \( x_i^* \).

Geometric meanings:
\[
\int_a^b g(x) \, dx = (\text{average value of } g \text{ on } [a, b])(b - a).
\]
\( g \geq 0 \Rightarrow \)
\[
\int_a^b g(x) \, dx = \text{area under graph of } g \text{ and over } [a, b].
\]
If \( g \) is the linear density (mass/length) of a wire over \([a, b]\),
\[
\int_a^b g(x) \, dx = \text{total mass of wire} \quad \& \quad \frac{\int_a^b x g(x) \, dx}{\int_a^b g(x) \, dx} = \text{center of mass}
\]
Integration over curves: Fix

- a curve $C \subset \mathbb{R}^2$
- parameterized by $r(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$, &
- a function $f : C \to \mathbb{R}$.

Divide $[a, b]$ into $n$ even subintervals $[t_{i-1}, t_i]$. Let $\Delta s_i$ be the length of $C$ between $r(t_{i-1})$ & $r(t_i)$.

Draw $[a, b]$ divided into five pieces. Draw curve and $r(t_i)$ for $i = 0, \ldots, 5$.

Pick any $t^*_i \in [t_{i-1}, t_i]$.

Define

$$\int_C f ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(r(t^*_i)) \Delta s_i,$$

if the limit exists & doesn’t depend on the $t^*_i$.

**Theorem:**

$$\int_C f ds = \int_{a}^{b} f(r(t))|r'(t)| dt$$

$$= \int_{a}^{b} f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$$

**Warning:** Don’t trace over the path more than once!
Proof.

\[ \Delta s_i \simeq |r(t_i) - r(t_{i-1})| \simeq |r'(t^*_i)|\Delta t \]

\[ \Rightarrow \int_C fds = \lim_{n \to \infty} \sum_{i=1}^{n} f(r(t^*_i)) \Delta s_i \]

\[ \simeq \lim_{n \to \infty} \sum_{i=1}^{n} f(r(t^*_i))|r'(t^*_i)| = \int_{a}^{b} f(r(t))|r'(t)|dt \]

Note: \( \int_C fds \) doesn’t depend on parameterization.
This also follows from substitution. We saw this when we calculated the length of a semi-circle last time.

Geometric meaning:

\[ \int_C fds = (\text{average value of } f \text{ on } C)(\text{length of } C) \]

\[ \Rightarrow \int_C 1ds = \text{length of } C \]

Question 2.

Let \( C \) be \( x^2 + y^2 = 1 \) & \( y \geq 0 \) & let \( f(x, y) = y \).
Use \( \int_C fds = (\text{average})(\text{length}) \) to guess \( \int_C fds \).

(A) 0  
(B) 1  
(C) 2  
(D) 3  
(E) 4  

Example.
Let \( C \) be \( x^2 + y^2 = 1 \) \& \( y \geq 0 \) \& let \( f(x, y) = y \).

\[
\mathbf{r}(t) = \langle \sin t, \cos t \rangle, \quad 0 \leq t \leq \pi
\]
\[
\mathbf{r}' = \langle \cos t, -\sin t \rangle
\]
\[
|\mathbf{r}'| = 1
\]

\[
\int_C f \, ds = \int_0^\pi f(\mathbf{r}(t))|\mathbf{r}'(t)| \, dt
\]
\[
= \int_0^\pi \sin t \cdot 1 \, dt
\]
\[
= -\cos t \bigg|_0^\pi = -(1 - 1) = 2
\]

If \( C \subset \mathbb{R}^3 \) is parameterized by \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \), \( a \leq t \leq b \), the situation is nearly identical:

\[
\int_C f \, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)| \, dt
\]
\[
= \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt
\]
More geometric meanings: $f \geq 0 \Rightarrow$

$\int_C f ds = \text{area of fence under graph of } f \text{ and over } C$.

If $f$ is the linear density of a wire shaped like $C$,

$\int_C f dx = \text{total mass of wire} \quad \&$

$(\bar{x}, \bar{y}) := \left( \frac{\int_C xf ds}{\int_C f ds}, \frac{\int_C yf ds}{\int_C f ds} \right) = \text{center of mass}.$

Why?

Let $\Delta m_i$ be the mass of the wire between $r(t_{i-1})$ & $r(t_i)$.

$$\text{Mass} = \text{density} \cdot \text{length} \Rightarrow \Delta m_i \simeq f(t^*_i) \Delta s_i \quad \text{for all } i \quad \Rightarrow$$

$$\text{Mass} = \sum_{i=1}^{n} \Delta m_i \simeq \sum_{i=1}^{n} f(t^*_i) \Delta s_i \quad \Rightarrow$$

$$\text{Mass} = \int_C f ds$$

The last question is optional.

**Question 3.** Assume there is a wire over $x^2 + y^2 = 1$ & $y \geq 0$ w/ constant density $\rho$.

Find it’s center of mass.

(A) (0, 2)

(B) (0, $\frac{2}{\pi}$)

(C) (2, 2)

(D) ($\frac{2}{\pi}$, $\frac{2}{\pi}$)

(E) (0, 1)