Prepare tabs:
   (1) lecture notes.
   (2) announcements.

Show tab 1.

PREVIOUSLY

Question 1.

Consider the disk \( D = x^2 + y^2 \leq 4 \).

Let \( f: D \rightarrow \mathbb{R} \) be continuous.

(A) \( f \) may not have a min.

(B) \( f \) has a min at a boundary point of \( D \).

(C) \( f \) has a min at a critical point.

(D) \( f \) has a min at a boundary or critical point.

(E) \( f \) has a min but it could be anywhere.

If you’re done: Find maximum and/or minimum.

Read tab 2 & read announcements.
Lagrange multiplies [14.8].

Example.
Find the max of $f(x, y) = x^2 - y^2$ on $D = x^2 + y^2 \leq 4$.

Critical points:
$\nabla f = (2x, -2y) = 0 \Rightarrow x = 0 \& y = 0$.

$\begin{align*}
f(0, 0) &= 0
\end{align*}$

Boundary points:
Method 1:
Parameterize $x^2 + y^2 = 4$ &
find critical points of $h(t) := f(x(t), y(t))$.

Method 2:

Draw axes & $x^2 + y^2 = 4$. Draw & label contour map of $f$.

By inspection: max of $f$ on $g(x, y) = 4$ is at $(2, 0) \& (-2, 0)$.

$\begin{align*}
f(2, 0) &= f(-2, 0) = 4
\end{align*}$

$\Rightarrow$ maximum is 4
Why?
Let $g(x, y) = x^2 + y^2$.

$(\sqrt{2}, \sqrt{2})$ is not a max.
Level curves of $f$ cut across circle $g = 4$.
$\iff$ can increase/decrease $f$ by moving along circle $(2, 0)$ is a max.
Level curves of $f$ are tangent to circle $g = 4$.
$\iff$ $\nabla f \& \nabla g$ are both perp to $x^2 + y^2 = 4$
$\implies$ $\nabla f \& \nabla g$ are parallel
$\implies$ $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$.

**Theorem** (Lagrange multipliers).
If $f(x_0, y_0)$ is the max value of $f$ on $g(x, y) = k$, either
- $\nabla g(x_0, y_0) = \langle 0, 0 \rangle$, or
- $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some $\lambda \in \mathbb{R}$.

Note:
If $f$ & $g$ are continuous $\implies g(x, y) = k$ is closed.
So if $g(x, y) = k$ is bounded max always exist.

Warning: need geometric argument if it isn’t bounded.
In practice:

1. Check $\nabla g \neq 0$ on $g(x, y) = k$ & $g(x, y) = k$ is bounded.
2. Find all $x, y$ & $\lambda$ s.t.
   - $\nabla f(x, y) = \lambda g(x, y)$ &
   - $g(x, y) = k$.
3. Calculate $f(x, y)$ for each $(x, y)$ from (2).
4. Pick the largest.

Example.
$f = x^2 - y^2$ & $g = x^2 + y^2$
$\Rightarrow \nabla f = \langle 2x, -2y \rangle$ & $\nabla g = \langle 2x, 2y \rangle$.
$\Rightarrow \nabla g \neq 0$ on $x^2 + y^2 = 4$.
$\& x^2 + y^2 = 4$ is bounded.

Solve $\nabla f = \lambda \nabla g$ & $g(x, y) = 4$

\[2x = \lambda 2x \quad \& \quad -2y = \lambda 2y \quad \& \quad x^2 + y^2 = 4 \iff \]
\[(x = 0 \ or \ \lambda = 1) \quad \& \quad (y = 0 \ or \ \lambda = -1) \quad \& \quad x^2 + y^2 = 1\]

$\Rightarrow$ 4 cases:
\begin{itemize}
  \item $x = 0 \ & \ y = 0 \ & \ x^2 + y^2 = 4 \Rightarrow \emptyset$
  \item $\lambda = 1 \ & \ \lambda = -1 \ & \ x^2 + y^2 = 4 \Rightarrow \emptyset$
  \item $x = 0 \ & \ \lambda = -1 \ & \ x^2 + y^2 = 4 \Rightarrow x = 0 \ & \ y = \pm 2.$
  \item $\lambda = 1 \ & \ y = 0 \ & \ x^2 + y^2 = 4 \Rightarrow x = \pm 2 \ & \ y = 0.$
\end{itemize}

\begin{center}
| $f(0, \pm 2) = -4$ & $f(\pm 2, 0) = 4$ |
\end{center}

$\Rightarrow$ Maximum is 4.
**Theorem (Lagrange multipliers).**

If \( f(x_0, y_0, z_0) \) is the max value of \( f \) on \( g(x, y, z) = k \), either

- \( \nabla g(x_0, y_0, z_0) = \langle 0, 0, 0 \rangle \), or
- \( \nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \) for some \( \lambda \in \mathbb{R} \).

Note: Similar thm’s hold for minimums.

**Example.** *Find the rectangular boxes of surface area 6 w/ largest and/or smallest volume.*

**Question 2.** There is...

(A) a largest box & a smallest box.
(B) a largest box but no smallest box.
(C) a smallest box but no largest box.
(D) neither a largest box nor a smallest box.

*Justify your answer.*

Length \( \ell \), width \( w \) & height \( h \)

Volume: \( f(\ell, w, h) = \ell wh \).

Surface area: \( g(\ell, w, h) = 2\ell h + 2\ell w + 2wh \).

If \( \geq 10 \) minutes, let students try to solve it themselves.

\[
\nabla f = \langle wh, \ell h, \ell w \rangle
\]
\[
\nabla g = \langle 2h + 2w, 2\ell + 2h, 2\ell + 2w \rangle
\]
Commen sense ⇒ ℓ, w & h are positive
⇒ ∇g does not vanish on g(x, y, z) = 6.

\[ \nabla g = \lambda \nabla f \quad \& \quad g(\ell, w, h) = 6 \quad \Leftrightarrow \]
\[ 2h + 2w = \lambda wh, \]
\[ 2\ell + 2h = \lambda \ell h, \]
\[ 2\ell + 2w = \lambda \ell w \quad \& \]
\[ 2\ell h + 2\ell w + 2wh = 6 \]
\[ \Leftrightarrow \frac{\lambda}{2} = \frac{1}{w} + \frac{1}{h} = \frac{1}{\ell} + \frac{1}{h} = \frac{1}{\ell} + \frac{1}{w} \quad \& \]
\[ 2\ell h + 2\ell w + 2wh = 6 \]
⇒ w = ℓ = h & 6\ell^2 = 6 ⇒ ℓ = ±1

Only 1 makes sense.

f(1, 1, 1) = 1.
⇒ Max is 1.