Question 1.

Classify the critical points of \( f(x, y) = x^2 - 2xy + y \).

(A) One local maximum.
(B) One local minimum.
(C) One saddle point.
(D) One local minimum and one local maximum.
(E) Two local maximums.

If you’re done: Find the absolute maximum and minimum.
**Absolute minima and maxima [14.7]**.

One variable:

\[ f : [p, q] \to \mathbb{R} \text{ continuous} \]

\[ \Rightarrow f \text{ has an absolute maximum at } a \in [p, q] \text{ s.t. either} \]

- \( a \) is a critical point, or
- \( a = p \) or \( q \).

**Warning:**

- False for \( f : \mathbb{R} \to \mathbb{R} \), e.g. \( f(x) = x \).
- False for \( f : (p, q) \to \mathbb{R} \), e.g. \( f(x) = x \).
- False if \( f \) is not continuous, e.g.

\[
f(x) = \begin{cases} 
\frac{1}{x} & x \neq 0 \\
0 & x = 0 
\end{cases}
\]

Graph \( f \).

Fix \( D \subset \mathbb{R}^2 \).

Draw a disk, a blob with a hole, and an infinite road.

**\( D \) is bounded** if it is contained in some disk.

Place checkmark/X near the bounded/unbounded regions.
(a, b) is a **boundary point** of D if every disk around (a, b) contains points in D and points that aren’t in D.

Draw region & P inside region, Q on boundary, and R outside it.

- Small disks around P only contain points in D
  ⇒ not in boundary.
- Small disks around R only contain points not in D
  ⇒ not in boundary.
- Any disk around Q contains points in D and not in D
  ⇒ in boundary.

A similar definition works for $D \subset \mathbb{R}$ or $\mathbb{R}^3$:

**Question 2.**

*How many points are in the boundary of $[p, q)$?*

(A) 0
(B) 1
(C) 2
(D) $\infty$

$D$ is **closed** if it contains all of its boundary points.
Example. \( x^2 + y^2 \leq 1 \)

Draw region.

The boundary \( x^2 + y^2 = 1 \) is in \( D \)
\( \Rightarrow \) closed.

Example. \( x^2 + y^2 < 1 \)

Draw region.

The boundary \( x^2 + y^2 = 1 \) isn’t in \( D \).
\( \Rightarrow \) not closed.

Example. \([p, q] \in \mathbb{R} \) are closed, but \((p, q) \notin [p, q)\) aren’t.

In practice: \( D \) is closed if it’s defined by only:

\[ f(x, y) \leq 0, \quad g(x, y) \geq 0 \quad \& \quad h(x, y) = 0. \]

(Not \( f(x, y) < 0 \) or \( g(x, y) > 0 \).)
Let
\[ D = \begin{cases} 
0 \leq x \leq 3 \\
0 \leq y \leq 2 
\end{cases} . \]

Draw \( D \).

**Question 3.** \( D \) is...

(A) closed and bounded.
(B) closed but not bounded.
(C) bounded but not closed.
(D) not closed or bounded.

Given \( D \subset \mathbb{R}^2 \) & \( f : D \to \mathbb{R} \):

\( f \) has an **absolute maximum** (**minimum**) at \((a, b)\) if 
\( f(x, y) \leq f(a, b) \) (\( f(x, y) \geq f(a, b) \)) for all \((x, y) \in D\).

**Theorem** (Extreme value theorem).
If \( f \) is continuous on \( D \subset \mathbb{R}^2 \) \& \( D \) is closed \& bounded
\[ \Rightarrow \] \( f \) has an absolute max (min) at \((a, b)\), where either
- \((a, b)\) is a critical point of \( f \), or
- \((a, b)\) is on the boundary of \( D \).
Why?
If $f$ has an absolute max at $(a, b)$ & $(a, b)$ is not on the boundary,
it must be a local max $\Rightarrow$ critical point.

In practice:
(1) find all possible $(a, b)$,
(2) compute $f(a, b)$ &
(3) pick the biggest.

**Example.** Find the absolute max & min of

$$f(x, y) = x^2 - 2xy + 2y$$
on $D = \begin{cases} 0 \leq x \leq 3 \\ 0 \leq y \leq 2 \end{cases}$

Draw $D$ & add points as you proceed.

**Critical points:**
We already found the critical point.

$(1, 1) \in D$

$$f(1, 1) = 1$$

**On left:**
$$g(y) := f(0, y) = 2y \quad 0 \leq y \leq 2.$$  
$$g'(y) = 2 \neq 0 \Rightarrow \text{no critical points.}$$
Endpoints:
\[ f(0, 0) = 0 \quad \& \quad f(0, 2) = 4 \]

On right:
\[ h(y) := f(3, y) = 9 - 6y + 2y = 9 - 4y \quad 0 \leq y \leq 2. \]
\[ h'(y) = -4 \neq 0 \Rightarrow \text{no critical points}. \]
Endpoints:
\[ f(3, 0) = 9 \quad \& \quad f(3, 2) = 1 \]

On bottom:
\[ G(x) = f(x, 0) = x^2 \quad 0 \leq x \leq 3. \]
\[ G'(x) = 2x = 0 \iff x = 0. \]
We already have \((0, 0)\) & endpoints.

On top:
\[ H(x) = f(x, 2) = x^2 - 4x + 2 \quad 0 \leq x \leq 3. \]
\[ H'(x) = 2x - 4 = 0 \iff x = 2 \]
\[ f(2, 2) = 2 \]
Can also argue from geometry.

Example. Find the point(s) on the graph \( z = \sqrt{x^2 + y^2} \) that are closest to and/or farthest from \((4, 2, 0)\).

Question 4.
(A) There’s a closest point & a farthest point.
(B) There’s a closest point but no farthest point.
(C) There’s a farthest point & but no closest point.
(D) There’s no closest point or farthest point.
Consider the distance squared from \((x, y, \sqrt{x^2 + y^2})\) to \((4, 2, 0)\):

\[
f(x, y) = (x - 4)^2 + (y - 2)^2 = (\sqrt{x^2 + y^2} - 0)^2 = x^2 - 8x + 16 + y^2 - 4y + 3 + x^2 + y^2 = 2x^2 - 2y^2 - 8x - 4y + 20
\]

\[
f_x = 4x - 8 = 0 \Rightarrow x = 2
\]
\[
f_y = 4y - 4 = 0 \Rightarrow y = 1
\]

\[
\Rightarrow z = \sqrt{2^2 + 1} = \sqrt{5}
\]

\[
\Rightarrow (2, 1, \sqrt{5}) \text{ is the only critical point}
\]
\[
\Rightarrow \text{it’s the closest point.}
\]