Game Theory and Duality

Let $A = \{a_{ij}\}$ be an $m \times n$ payoff matrix for a game with zero sum. If the first player chooses his/her strategy i with probability x_i for every $i = 1, \dots, n$, and the second player chooses his/her strategy j with probability y_j for all $j = 1, \dots, m$ then the expectation of the profit of the first player will be

$$F(A, \mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} y_i x_j = \mathbf{y}^T A \mathbf{x}.$$

Thus the first player can provide the expected *profit*

$$v_1(A) = \max_{\mathbf{x}} \min_{\mathbf{y}} F(A, \mathbf{x}, \mathbf{y}),$$

and the second player's expected *loss* can be made at most

$$v_2(A) = \min_{\mathbf{y}} \max_{\mathbf{x}} F(A, \mathbf{x}, \mathbf{y}).$$

It is not hard to see that $v_1(A) \le v_2(A)$ for every payoff matrix A.

Theorem 1 *For every payoff matrix A, $v_1(A) = v_2(A)$.*

PROOF. Consider the following LP1:

$$\text{Find} \quad \min -\mathbf{v_1}$$

$$\text{such that} \quad \begin{array}{cccccccc} a_{1,1}x_1 & +a_{1,2}x_2 & \dots & +a_{1,n}x_n & -v_1 & \ge & 0 & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ a_{m,1}x_1 & +a_{m,2}x_2 & \dots & +a_{m,n}x_n & -v_1 & \ge & 0 & \\ -x_1 & -x_2 & \dots & -x_n & & = & -1 & \\ & & & & x_i & \ge & 0 & \forall i \end{array}$$

One can check that the maximum possible v_1 in this LP is exactly $v_1(A)$. The reason for it is that the inequalities mean that with the choice $(x_1, \dots, x_m)$ of the probabilities, the first player can get at least v_1 against every pure strategy of the second player. But then he can guarantee this gain against every mixed strategy as well.

Similarly, $v_2(A)$ is the solution of the the following LP2:

$$\text{Find} \quad \max -\mathbf{v_2}$$

$$\text{such that} \quad \begin{array}{cccccccc} a_{1,1}y_1 & +a_{2,1}y_2 & \dots & +a_{m,1}y_m & -v_2 & \le & 0 & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ a_{1,m}y_1 & +a_{2,m}y_2 & \dots & +a_{n,m}y_m & -v_2 & \le & 0 & \\ -y_1 & -y_2 & \dots & -y_m & & = & -1 & \\ & & & & x_j & \ge & 0 & \forall j \end{array}$$

Both these problems have feasible solutions (any pure strategies would do). Moreover, they are DUAL. This proves the theorem.